STABLE SPLITTINGS, SPACES OF REPRESENTATIONS AND ALMOST COMMUTING ELEMENTS IN LIE GROUPS

ALEJANDRO ADEM*, FREDERICK R. COHEN**, AND JOSÉ MANUEL GÓMEZ

Abstract. In this paper the space of almost commuting elements in a Lie group is studied through a homotopical point of view. In particular a stable splitting after one suspension is derived for these spaces and their quotients under conjugation. A complete description for the stable factors appearing in this splitting is provided for compact connected Lie groups of rank one.By using symmetric products, the colimits $\mathrm{Rep}(\mathbb{Z}^n, SU)$, $\mathrm{Rep}(\mathbb{Z}^n, U)$ and $\mathrm{Rep}(\mathbb{Z}^n, Sp)$ are explicitly described as finite products of Eilenberg-MacLane spaces.

1. Introduction

Let G denote a Lie group, for each integer $n \geq 1$, the set $\mathrm{Hom}(\mathbb{Z}^n, G)$ can be topologized in a natural way as a subspace of G^n and corresponds to the space of ordered commuting n-tuples in G. The group G acts by conjugation on $\mathrm{Hom}(\mathbb{Z}^n, G)$ and the orbit space $\mathrm{Rep}(\mathbb{Z}^n, G) := \mathrm{Hom}(\mathbb{Z}^n, G)/G$ can be identified with the moduli space of isomorphism classes of flat connections on principal G-bundles over the torus $(\mathbb{S}^1)^n$. Now if $H = G/K$, where $K \subset G$ is a closed subgroup contained in the center of G, then a commuting n-tuple $(x_1, ..., x_n)$ of elements in H, lifts to a sequence $(\tilde{x}_1, ..., \tilde{x}_n)$ of elements in G which do not necessarily commute; instead they form a K*–almost commuting sequence* in G in the sense that $[\tilde{x}_i, \tilde{x}_j] \in K$ for all i and j.

Denote the space of all K-almost commuting n-tuples in G by $B_n(G, K)$; note that it fits into a principal K^n–bundle $B_n(G, K) \to \mathrm{Hom}(\mathbb{Z}^n, G/K)$. The group G acts by conjugation on $B_n(G, K)$ with orbit space denoted by $\bar{B}_n(G, K) = B_n(G, K)/G$. From the standard classification theorems (see [17, Theorem 6.19]), it is known that if G is a compact, connected Lie group, then $G = \tilde{G}/K$, where K is a finite subgroup in the center of $\tilde{G}$ and $\tilde{G} = (\mathbb{S}^1)^r \times G_1 \times \cdots \times G_s$ where the $G_1, \ldots, G_s$ are simply connected simple Lie groups. Thus the K–almost commuting elements clearly play a key role in the analysis of spaces of commuting elements for compact Lie groups. This approach was applied in [11] for the more geometrically accessible situation of commuting triples. This paper is concerned with the study of the spaces of the form $B_n(G, K)$ and their orbit spaces $\bar{B}_n(G, K)$ from a *homotopical* point of view for all values of $n \geq 1$. In particular a basic goal is to compute the number of path-connected components and determine their stable homotopy types.

A basic ingredient is the existence of a natural simplicial structure on $B_*(G, K)$ which provides a filtration for each $B_n(G, K)$ given by the degeneracy maps

$$\{(1_G, ..., 1_G)\} = S^n_n(G, K) \subset S^{n-1}_n(G, K) \subset \cdots \subset S^0_n(G, K) = B_n(G, K),$$

*Partially supported by NSERC.
**Partially supported by the NSF.

where $S_n^r(G,K)$ is the subspace of $B_n(G,K)$ of sequences $(x_1,...,x_n)$ where at least r of the x_i equal 1_G. It turns out that when G is compact the pair $(S_n^r(G,K), S_n^{r+1}(G,K))$ is a G–equivariant NDR pair for $0 \leq r \leq n$. This technical condition implies (see [2], [3]) that the previous filtration splits after one suspension and thus the following theorem is obtained

Theorem 1.1. *Suppose that G is a compact Lie group and $K \subset Z(G)$ is a closed subgroup. For all $n \geq 0$ there is a natural G-equivariant homotopy equivalence*

$$\Sigma(B_n(G,K)) \simeq \bigvee_{1\leq r\leq n} \Sigma\left(\bigvee^{\binom{n}{r}} B_r(G,K)/S_r^1(G,K)\right).$$

In particular, there is a natural homotopy equivalence

$$\Sigma(\bar{B}_n(G,K)) \simeq \bigvee_{1\leq r\leq n} \Sigma\left(\bigvee^{\binom{n}{r}} \bar{B}_r(G,K)/\bar{S}_r^1(G,K)\right).$$

A particularly important case of Theorem 1.1 occurs when $K = \{1_G\}$. In this case, the theorem agrees with [3, Theorem 1.6] for compact Lie groups with the important additional property that the splitting map is a G-equivariant homotopy equivalence. This defines a natural homotopy equivalence between the quotient spaces modulo conjugation, thus yielding a new decomposition at the level of spaces of representations:

Theorem 1.2. *For G a compact Lie group, there is a homotopy equivalence*

$$\Sigma Rep(\mathbb{Z}^n, G) \simeq \bigvee_{1\leq r\leq n} \Sigma\left(\bigvee^{\binom{n}{r}} Rep(\mathbb{Z}^n, G)/\bar{S}_r^1(G)\right).$$

When $\mathrm{Rep}(\mathbb{Z}^n, G)$ is path-connected for all $n \geq 1$ then the following theorem completely describes this decomposition:

Theorem 1.3. *Let G be a compact, connected Lie group such that $Rep(\mathbb{Z}^n, G)$ is connected for every $n \geq 1$. Let T be a maximal torus of G and W the Weyl group associated to T. Then*

$$Rep(\mathbb{Z}^n, G) \cong T^n/W$$

with W acting diagonally on T^n. Moreover there is a homotopy equivalence

$$\Sigma Rep(\mathbb{Z}^n, G) \simeq \bigvee_{1\leq r\leq n} \Sigma(\bigvee^{\binom{n}{r}} T^{\wedge r}/W),$$

where $T^{\wedge r}$ is the r-fold smash product of T and W acts diagonally on $T^{\wedge r}$.

The previous theorem applies in particular to the cases $G = U(m)$, $Sp(m)$ and $SU(m)$. It turns out that for such G the representation spaces $\mathrm{Rep}(\mathbb{Z}^n, G)$ can be identified in terms of symmetric products.

Proposition 1.4. *There are homeomorphisms*

$$\begin{aligned} Rep(\mathbb{Z}^n, U(m)) &\cong SP^m((\mathbb{S}^1)^n), \\ Rep(\mathbb{Z}^n, Sp(m)) &\cong SP^m((\mathbb{S}^1)^n/\mathbb{Z}/2), \end{aligned}$$

where $\mathbb{Z}/2$ acts diagonally by complex conjugation on each factor. Moreover for each $m, n \geq 1$ the determinant map defines a locally trivial bundle

$$Rep(\mathbb{Z}^n, SU(m)) \to Rep(\mathbb{Z}^n, U(m)) \to (\mathbb{S}^1)^n.$$

This identification provides a number of interesting consequences, for example

Corollary 1.5. *For every $m \geq 1$ there are homeomorphisms*

$$Rep(\mathbb{Z}^2, Sp(m)) \cong \mathbb{C}P^m,$$

$$Rep(\mathbb{Z}^2, SU(m)) \cong \mathbb{C}P^{m-1}.$$

Another consequence of Proposition 1.4 is the determination of the homotopy type of the inductive limits of the spaces of commuting elements modulo conjugation for the (special) unitary groups and the symplectic groups. Note that this result for the unitary groups is also proved in the recent preprint [22].

Theorem 1.6. *For every $n \geq 1$ there are homotopy equivalences*

$$\begin{aligned} Rep(\mathbb{Z}^n, SU) &\simeq \prod_{2 \leq i \leq n} K(\mathbb{Z}^{\binom{n}{i}}, i), \\ Rep(\mathbb{Z}^n, U) &\simeq \prod_{1 \leq i \leq n} K(\mathbb{Z}^{\binom{n}{i}}, i), \\ Rep(\mathbb{Z}^n, Sp) &\simeq \prod_{1 \leq i \leq \lfloor n/2 \rfloor} K(\mathbb{Z}^{\binom{n}{2i}} \oplus (\mathbb{Z}/2)^{r(2i)}, 2i), \end{aligned}$$

where $r(i)$ is an integer given by

$$r(i) = \begin{cases} \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n-i-1} & \text{if} \quad 1 \leq i \leq n \\ 0 & \text{if} \quad i > n. \end{cases}$$

Also, explicit computations for the stable factors that appear in Theorem 1.1 are provided for non-abelian compact connected rank one Lie groups. If G is such a group then it is isomorphic to $SU(2)$ or $SO(3)$. Let λ_n denote the canonical vector bundle over the projective space $\mathbb{R}P^n$ and for a vector bundle μ over $\mathbb{R}P^n$, $(\mathbb{R}P^n)^\mu$ denotes the corresponding Thom space. In these cases the factors $B_n(G, K)/S_n^1(G, K)$ are given as follows.

$$\mathrm{Hom}(\mathbb{Z}^n, SO(3))/S_n^1(SO(3)) \cong \begin{cases} \mathbb{R}P^3 & \text{if } n = 1, \\ (\mathbb{R}P^2)^{n\lambda_2} \vee \left(\bigvee_{C(n)} (\mathbb{S}^3/Q_8)_+ \right) & \text{if } n \geq 2, \end{cases}$$

where Q_8 is the quaternion group of order 8 and

$$C(n) = \frac{1}{2}(3^{n-1} - 1).$$

For $SU(2)$, there are two possibilities as $Z(SU(2)) \cong \mathbb{Z}/2$:

$$\mathrm{Hom}(\mathbb{Z}^n, SU(2))/S_n^1(SU(2)) \cong \begin{cases} \mathbb{S}^3 & \text{if } n = 1, \\ (\mathbb{R}P^2)^{n\lambda_2}/s_n(\mathbb{R}P^2) & \text{if } n \geq 2. \end{cases}$$

Here s_n is the zero section of the vector bundle $n\lambda_2$. The other case is the space of almost commuting elements in $SU(2)$:

$$B_n(SU(2), \mathbb{Z}/2)/S_n^1(SU(2), \mathbb{Z}/2) \cong \begin{cases} \mathbb{S}^3 & \text{if } n = 1, \\ (\bigvee_{K(n)} \mathbb{R}P^3_+) \vee (\mathbb{R}P^2)^{n\lambda_2}/s_n(\mathbb{R}P^2) & \text{if } n \geq 2, \end{cases}$$

where

$$K(n) = \frac{7^n}{24} - \frac{3^n}{8} + \frac{1}{12}.$$

Finally, using the work in [11], explicit descriptions for the stable factors modulo conjugation $\bar{B}_n(G, K)/\bar{S}_n^1(G, K)$ that appear in Theorem 1.1 are given when G is a compact, connected, simply connected, simple Lie group and for $n = 1$, 2 and 3.

2. The simplicial spaces $B_n(q, G, K)$

A family of simplicial spaces $B_n(q, G, K)$ is defined in this section, where G is a Lie group and $K \subset Z(G)$ is a closed subgroup. These spaces form a generalization of the simplicial spaces $B_n(q, G)$ defined in [6] as the case $K = \{1_G\}$ corresponds to $B_n(q, G)$. Some general properties are derived.

The definition of the spaces $B_n(q, G)$ is given first. Let F_n be the free group on n-letters. For each $q \geq 1$, let Γ^q denote the q-stage of the descending central series for F_n. The groups Γ^q are defined inductively by setting $\Gamma^1 = F_n$ and $\Gamma^{q+1} = [\Gamma^q, F_n]$. As in [6] define

$$B_n(q, G) := \mathrm{Hom}(F_n/\Gamma^q, G).$$

The topology on $B_n(q, G)$ is given as follows. An element in $\mathrm{Hom}(F_n/\Gamma^q, G)$ can be seen as a homomorphism $f : F_n \to G$ such that $f(\Gamma^q) = \{1_G\}$ so that f descends to a homomorphism $\bar{f} : F_n/\Gamma^q \to G$. The map f is determined by the image of a set of generators $e_1, ..., e_n$; that is, f is determined by the n-tuple $(f(e_1), ..., f(e_n))$. This gives an inclusion of $\mathrm{Hom}(F_n/\Gamma^q, G)$ into G^n and via this inclusion $B_n(q, G)$ is given the subspace topology. The different $B_n(q, G)$ form simplicial spaces and can be used to obtain a filtration

$$B(2, G) \subset B(3, G) \subset \cdots \subset B(q, G) \subset \cdots B(\infty, G) = BG,$$

of BG, where $B(q, G)$ denotes the geometric realization of $B_n(q, G)$.

Definition 2.1. Let Q be a group. A sequence of subgroups $\Gamma^r(Q)$ is defined inductively as follows. For $r = 1$, $\Gamma^1(Q) = Q$ and $\Gamma^{i+1}(Q) = [\Gamma^i(Q), Q]$. The different $\Gamma^r(Q)$ form the descending central series of Q

$$\cdots \subset \Gamma^{i+1}(Q) \subset \Gamma^i(Q) \subset \cdots \Gamma^2(Q) \subset \Gamma^1(Q) \subset Q.$$

A discrete group Q is said to be nilpotent if $\Gamma^{m+1}(Q) = \{1\}$ for some m and the least such m is called the nilpotency class of Q. Suppose that $\bar{f} : F_n/\Gamma^q \to G$ is a homomorphism induced by $f : F_n \to G$. If $Q = \mathrm{Im} f$, then $f(\Gamma^q) = \Gamma^q(Q) = \{1_G\}$; that is, Q is nilpotent of nilpotency

class less than q. Note that if $q \geq 2$ and if G is a topological group of nilpotency less class than q, then $B(q, G) = BG$.

Suppose now that H is a Lie group of the form G/K, for a closed subgroup $K \subset Z(G)$; that is, suppose that H fits into a central extension of Lie groups

$$1 \to K \to G \to H \to 1.$$

Take $\bar{f} : F_n/\Gamma^q \to H$ an element in $B_n(q, H)$ induced by a homomorphism $f : F_n \to H$. Then f is determined by the n-tuple $(x_1, ..., x_n) = (f(e_1), ..., f(e_n))$ of elements in H. Since $K \subset Z(G)$ is a closed subgroup, the natural map $p : G \to G/K$ is both a homomorphism and a principal K-bundle. For each $1 \leq i \leq n$, take $\tilde{x}_i$ a lifting of x_i in G. The elements $\tilde{x}_1, ..., \tilde{x}_n$ define a map $g : F_n \to G$. Note however that this map does not necessarily descend to a map $F_n/\Gamma^q \to G$. Instead, $p(g(\Gamma^q)) = f(\Gamma^q) = \{1\} \in G/K$ and it follows that $g(\Gamma^q) \subset K$. This motivates the following definition.

Definition 2.2. Let G be a Lie group, $K \subset Z(G)$ a closed subgroup and $n, q \geq 1$. Define

$$B_n(q, G, K) = \{f : F_n \to G \mid f \text{ is a homomorphism and } f(\Gamma^q) \subset K\}.$$

As in the case of $B_n(q, G)$, the set $B_n(q, G, K)$ is topologized as a subspace of G^n. Note that if $K = \{1_G\}$ is the trivial subgroup, then $B_n(q, G) = B_n(q, G, \{1_G\})$ and when $q = 2$, the space $B_n(2, G, K)$ consists of the n-tuples $(x_1, ..., x_n)$ in G such that $[x_i, x_j] \in K$ for all i and j.

Lemma 2.3. *Suppose that G is a Lie group, $K \subset Z(G)$ a closed subgroup and $q \geq 1$. Then the restriction of the natural map $p^n : G^n \to (G/K)^n$ to $B_n(q, G, K)$ defines a G-equivariant $Hom(F_n/\Gamma^q, K) \cong K^n$-principal bundle*

$$\phi_n : B_n(q, G, K) \to B_n(q, G/K).$$

Proof: The natural map $p : G \to G/K$ is a principal K-bundle and since K is central, the conjugation action of G induces an action of G on G/K such that p is G-equivariant. It follows that $p^n : G^n \to (G/K)^n$ is a G-equivariant principal K^n-bundle. By restriction, the map

$$\phi_n = (p^n)_{|(p^n)^{-1}(B_n(q,G/K))} : (p^n)^{-1}(B_n(q, G, K)) \to B_n(q, G/K)$$

is also a principal K^n-bundle. Thus it is only necessary to show that

$$(p^n)^{-1}(B_n(q, G/K)) = B_n(q, G, K).$$

From the definition it is clear that $B_n(q, G, K) \subset (p^n)^{-1}(B_n(q, G/K))$. On the other hand, take $(x_1, ..., x_n) \in G^n$ such that $(p(x_1), ..., p(x_n)) \in B_n(q, G)$. This is equivalent to saying that the subgroup $\bar{Q} \subset G/K$ generated by $p(x_1), ..., p(x_n)$ has nilpotency class less than q. If Q if the subgroup of G generated by $x_1, ..., x_n$, then $p(\Gamma^q Q) = \Gamma^q(p(Q)) = \Gamma^q(\bar{Q}) = \{1_G\}$. It follows that $\Gamma^q Q \subset K$ and thus $(x_1, ..., x_n) \in B_n(q, G, K)$. □

Remark. In [15, Lemma 2.2], Goldman showed that if π is a finitely generated group and $p : G' \to G$ is a local isomorphism, then composition with p defines a continuous map

$$p_* : \mathrm{Hom}(\pi, G') \to \mathrm{Hom}(\pi, G),$$

such that the image of p_* is a union of connected components of $\mathrm{Hom}(\pi, G)$ and if Q is a connected component in the image of p_*, then the restriction

$$(p_*)_{|p_*^{-1}(Q)} : p_*^{-1}(Q) \to Q$$

is a covering space with covering group $\mathrm{Hom}(\pi, K)$, where $K = \mathrm{Ker}(p)$. In particular, this applies for $\pi = F_n/\Gamma_q$. This shows that in this case p defines a continuous map

$$p_* : B_n(q, G') \to B_n(q, G),$$

such that the image of p_* is a union of connected components of $B_n(q, G)$. However, the map p_* is not surjective in general. For example, if $G' = SU(2)$ and $G = G'/K$, where $K = Z(G) = \{\pm 1\}$, then $G'/K = SU(2)/\{\pm 1\} \cong SO(3)$, $B_2(2, SU(2)) = \mathrm{Hom}(\mathbb{Z}^2, SU(2))$ is path-connected but $B_2(2, SO(3)) = \mathrm{Hom}(\mathbb{Z}^2, SO(3))$ has two path-connected components. The lack of surjectivity of p_* is precisely what motivates the study of the spaces $B_n(q, G, K)$.

Take now G a Lie group and $K \subset Z(G)$ a closed subgroup. For each $q \geq 2$ the collection $B_*(q, G, K)$ can be seen as a simplicial space. Indeed, recall that for any Lie group G there is an associated simplicial space B_*G, where $B_nG = G^n$,

$$s_j(g_1, ..., g_n) = (g_1, ..., g_j, 1_G, g_{j+1}, ..., g_n),$$

and

$$\partial_i(g_1, ...g_n) = \begin{cases} (g_2, ..., g_n) & \text{if } i = 0, \\ (g_1, ..., g_i g_{i+1}, ..., g_n) & \text{if } 0 < i < n, \\ (g_1, ..., , g_{n-1}) & \text{if } i = n. \end{cases}$$

The restriction of the maps s_j and ∂_i keep the different $B_n(q, G, K)$ invariant as the maps s_j and ∂_i are induced by homomorphisms between free groups and if $f : A \to B$ is a homomorphism of groups, then $f(\Gamma^q(A)) \subset \Gamma^q(B)$. Note that the conjugation action of G on each $B_n(q, G, K)$ makes $B_*(q, G, K)$ into a G–simplicial space. Moreover, the different maps

$$\phi_n : B_n(q, G, K) \to B_n(q, G/K)$$

define a G-equivariant simplicial map.

The spaces of the form $B_n(2, G, K)$ are the subject of study of this article and from now on only the case $q = 2$ will be considered. The abbreviated notation $B_n(G, K)$ from the introduction will be used from now on.

3. Almost commuting elements in Lie groups

In this section, following the approach in [11], a decomposition for the spaces $B_n(G, K)$ is given by keeping track of the different commutators. This decomposition gives a good way of detecting connected components for the space $\mathrm{Hom}(\mathbb{Z}^n, G/K)$.

Suppose that G is a Lie group and that $K \subset Z(G)$ is a closed subgroup. By definition $B_n(G, K)$ is the space of homomorphisms $f : F_n \to G$ such that $f(\Gamma^2) \subset K$. Since $\Gamma^2 = [F_n, F_n]$ it follows that under the identification of $B_n(G, K)$ as a subspace of G^n, an n-tuple $\underline{x} := (x_1, ..., x_n) \in G^n$ belongs to $B_n(G, K)$ if and only if $d_{ij} := [x_i, x_j] \in K$; that is, the elements x_i and x_j commute up to elements in K. Such a sequence is called an almost commuting sequence.

Definition 3.1. Suppose G is a Lie group and $K \subset Z(K)$ is a subgroup. A sequence $(x_1, ..., x_n) \in G$ is said to be K-almost commuting if $[x_i, x_j] \in K$ for all i and j.

According to the previous discussion, $B_n(G,K)$ is precisely the space of K-almost commuting n-tuples in G. Take $\underline{x} = (x_1, ..., x_n) \in B_n(G,K)$, therefore $d_{ij} = [x_i, x_j] \in K$. The elements d_{ij} are such that $d_{ii} = 1$ and $d_{ij} = d_{ji}^{-1}$, this means that the matrix $D = (d_{ij})$, which is called the type of $\underline{x}$, is an antisymmetric matrix with entries in $K \subset Z(G)$. Clearly the entries $d_{ij} = d_{ij}(\underline{x})$ are continuous functions of $\underline{x}$ with values in K. A decomposition of the space $B_n(G,K)$ is obtained by keeping track of the different commutators d_{ij}. More precisely, let $\pi_0 : K \to \pi_0(K)$ be the map that identifies the connected components of elements in K and define $T(n, \pi_0(K))$ to be the set of all $n \times n$ antisymmetric matrices $C = (c_{ij})$ with entries in $\pi_0(K)$. Given $C \in T(n, \pi_0(K))$ define

$$\mathcal{AC}_G(C) = \{(x_1, ..., x_n) \in G^n \mid \pi_0([x_i, x_j]) = c_{ij}\};$$

that is, $\mathcal{AC}_G(C)$ is the set of almost commuting sequences in G of type D with $\pi_0(D) = C$. Each $\mathcal{AC}_G(C)$ is a subspace of $B_n(G,K)$ that is both open and closed because $[*, *]$ is a continuous function and $\pi_0(K)$ is discrete. It follows that each $\mathcal{AC}_G(C)$ is a union of connected components of $B_n(G,K)$. In addition, the conjugation action of G on G^n, restricts to an action of G on each $\mathcal{AC}_G(C)$. Indeed, if $[x_i, x_j] = d_{ij} \in Z(G)$, then $[gx_ig^{-1}, gx_jg^{-1}] = gd_{ij}g^{-1} = d_{ij}$. Following [11] the orbit space of $\mathcal{AC}_G(C)$ under this action is denoted by

$$\mathcal{M}_G(C) := \mathcal{AC}_G(C)/G.$$

Note that $(Z(G))^n$ acts on each $B_n(G,K)$ by left componentwise multiplication. Also, when $K = \{1_G\}$ the space $B_n(G, \{1_G\})$ is precisely $\mathrm{Hom}(\mathbb{Z}^n, G)$. The orbit space under the action of G is denoted by

$$\bar{B}_n(G,K) := B_n(G,K)/G$$

and when $K = \{1_G\}$

$$\mathrm{Rep}(\mathbb{Z}^n, G) := \mathrm{Hom}(\mathbb{Z}^n, G)/G.$$

By definition, the different $\mathcal{AC}_G(C)$ give a decomposition of $B_n(G,K)$ into subspaces that are both open and closed as follows:

$$B_n(G,K) = \bigsqcup_{C \in T(n, \pi_o(K))} \mathcal{AC}_G(C). \tag{1}$$

The natural map $p : G \to G/K$ is an open map and the restriction of p^n to $B_n(G,K)$ is a principal K^n-bundle $\phi_n : B_n(G,K) \to \mathrm{Hom}(\mathbb{Z}^n, G/K)$ by Lemma 2.3 and is also an open map. This shows that if $C \in T(n, \pi_0(K))$, then

$$\mathrm{Hom}(\mathbb{Z}^n, G/K)_C := p^n(\mathcal{AC}_G(C))$$

is an open subset of $\mathrm{Hom}(\mathbb{Z}^n, G/K)$ and (1) induces a decomposition

$$\mathrm{Hom}(\mathbb{Z}^n, G/K) = \bigsqcup_{C \in T(n, \pi_0(K))} \mathrm{Hom}(\mathbb{Z}^n, G/K)_C. \tag{2}$$

The complement of each $\mathrm{Hom}(\mathbb{Z}^n, G/K)_C$ is open since it is a union of spaces of the form $\mathrm{Hom}(\mathbb{Z}^n, G/K)_{C'}$, therefore each $\mathrm{Hom}(\mathbb{Z}^n, G/K)_C$ is both open and closed in $\mathrm{Hom}(\mathbb{Z}^n, G/K)$ hence a union of connected components. Moreover, for each element $C \in T(n, \pi_o(K))$ the natural map

$$\mathcal{AC}_G(C) \to \mathrm{Hom}(\mathbb{Z}^n, G/K)_C$$

is a principal K^n-bundle by Lemma 2.3. The decomposition (2) gives a good way of detecting connected path components of the spaces of the type $\mathrm{Hom}(\mathbb{Z}^n, G/K)$. Let $\mathcal{N}(G,K)$ be the number of different $C \in T(n, \pi_0(K))$ for which $\mathcal{AC}_G(C)$ is nonempty. Then the following is an immediate corollary.

Corollary 3.2. *Suppose that G is a Lie group and $K \subset Z(G)$ is a closed subgroup. Then $Hom(\mathbb{Z}^n, G/K)$ has at least $\mathcal{N}(G,K)$ connected components.*

Example 3.3. $SO(m)$ fits into a central extension of Lie groups

$$1 \to \mathbb{Z}/2 \xrightarrow{i} Spin(m) \xrightarrow{p} SO(m) \to 1.$$

Composition with p gives an exact sequence of pointed sets

$$1 \to \mathrm{Hom}(\mathbb{Z}^n, \mathbb{Z}/2) \to \mathrm{Hom}(\mathbb{Z}^n, Spin(m)) \xrightarrow{p_*} \mathrm{Hom}(\mathbb{Z}^n, SO(m)).$$

In this case p_* is not surjective and the image of p_* does not contain all the path-connected components of $\mathrm{Hom}(\mathbb{Z}^n, SO(m))$. As in (2) there is a decomposition

$$\mathrm{Hom}(\mathbb{Z}^n, SO(m)) = \bigsqcup_{C \in T(n,\mathbb{Z}/2)} \mathrm{Hom}(\mathbb{Z}^n, SO(m))_C, \tag{3}$$

where each $\mathrm{Hom}(\mathbb{Z}^n, SO(m))_C$ is a union of connected components of $\mathrm{Hom}(\mathbb{Z}^n, SO(m))$. Given a homomorphism $f : \mathbb{Z}^n \to SO(m)$, it lifts compatibly to a central extension

$$1 \to \mathbb{Z}/2 \to \Gamma \to \mathbb{Z}^n \to 1$$

which maps to $Spin(m)$. Such an extension is determined by an element in $H^2(\mathbb{Z}^n, \mathbb{Z}/2) \cong (\mathbb{Z}/2)^{\binom{n}{2}}$. Fixing a basis $x_1, \ldots, x_n$ for $\mathbb{Z}^n$, the components for this cohomology class are determined by taking liftings $\tilde{x_i}$ and $\tilde{x_j}$ of x_i and x_j respectively and calculating their commutator. This is how the transgression map $H^1(\mathbb{Z}/2, \mathbb{Z}/2) \to H^2(\mathbb{Z}^n, \mathbb{Z}/2)$ associated to the extension Γ is computed in group cohomology. The values obtained are thus the same as those used to define the matrix $C \in T(n, \mathbb{Z}/2)$ associated to the almost commuting n–tuple $\tilde{x_1}, \ldots, \tilde{x_n}$; moreover, note that the two sets $H^2(\mathbb{Z}^n, \mathbb{Z}/2)$ and $T(n, \mathbb{Z}/2)$ have the same cardinality. By construction the 2–dimensional cohomology class associated to Γ must be the pullback of the cohomology class which determines $Spin(m)$; this is precisely the second Stiefel–Whitney class. More precisely, given $f : \mathbb{Z}^n \to SO(m)$, the class $w_2(f) \in H^2(\mathbb{Z}^n, \mathbb{Z}/2)$ is defined as $f^*(w_2)$; where $w_2 \in H^2(BSO(m), \mathbb{Z}/2)$ is the universal Stiefel–Whitney class.

What this shows is that the decomposition of $\mathrm{Hom}(\mathbb{Z}^n, SO(m))$ described above is the decomposition determined by the second Stiefel–Whitney class; in other words f and g are in the same C–component if and only if $w_2(f) = w_2(g)$. Moreover in [4] it was proved that for $m >> 0$ all the values of w_2 can be attained by suitable homomorphisms, thus establishing that $\mathrm{Hom}(\mathbb{Z}^n, SO(m))$ has at least $\mathcal{N}(Spin(m), \mathbb{Z}/2) = 2^{\binom{n}{2}}$ non–empty components. Note that the component corresponding to the trivial cohomology class is precisely $p_*(\mathrm{Hom}(\mathbb{Z}^n, Spin(m)))$, i.e. those homomorphisms that can be lifted to $Spin(m)$ (which happens if and only if $w_2 = 0$).

Example 3.4. In [5] the authors use the decomposition (2) to study the spaces of the form $\mathrm{Hom}(\mathbb{Z}^n, G_{m,p})$, where for a prime number p and an integer $m \geq 1$, $G_{m,p}$ denotes a central

product of m-copies of $SU(p)$. There it is shown that $\mathrm{Hom}(\mathbb{Z}^n, G_{m,p})$ has

$$N(n,m,p) = \frac{p^{(m-1)(n-2)}(p^n-1)(p^{n-1}-1)}{p^2-1} + 1$$

path-connected components of which $N(n,m,p)-1$ are homeomorphic to

$$(SU(p))^m/((\mathbb{Z}/p)^{m-1} \times E_p),$$

where $E_p \subset SU(p)$ is the quaternion group Q_8 of order eight when $p=2$ and the extraspecial p–group of order p^3 and exponent p when $p>2$.

4. NDR pairs and compact Lie groups

In this section it is shown that if G is a compact Lie group and $K \subset Z(G)$ is a closed subgroup, then the simplicial spaces $B_*(G,K)$ are simplicially NDR as defined in [2].

To start the usual definitions of G-NDR pairs and strong NDR pairs are given.

Definition 4.1. Suppose that G is a topological group and take (X,A) a G-pair. The pair (X,A) is said to be a G-NDR pair if there exists a G-equivariant continuous function

$$h : X \times [0,1] \to X,$$

with G acting trivially on $[0,1]$ and a G-invariant continuous map $u : X \to [0,1]$ such that the following conditions are satisfied.

(1) $A = u^{-1}(0)$,
(2) $h(x,0) = x$ for all $x \in X$,
(3) $h(a,t) = a$ for all $a \in A$ and all $t \in [0,1]$, and
(4) $h(x,1) \in A$ for all $x \in u^{-1}([0,1))$.

Note that when $G = \{1\}$ is the trivial group then the previous definition corresponds to Steenrod's original definition of an NDR pair as in [24]. The pair (h,u) is called a G-NDR representation for the topological pair (X,A). The following is a straightforward lemma.

Lemma 4.2. *Suppose that (X,A) is a G-NDR pair represented by (h,u). By passing to orbit spaces, the pair (h,u) induces an NDR representation $(\bar{h},\bar{u})$ for the pair $(X/G, A/G)$. Also, for any subgroup $H \subset G$, by passing to fixed points (h,u) induces an NDR representation (h^H,u^H) for the pair (X^H,A^H).*

In [19], May considered a more restricted version of NDR pairs to study simplicial spaces, in particular to study the natural filtration of the geometric realization of a simplicial space. The precise definition is the following.

Definition 4.3. An NDR pair (X,A) represented by the pair (h,u) is said to be a strong NDR pair if $u(h(x,t)) < 1$ for all $t \in [0,1]$, whenever $u(x) < 1$. A G-NDR pair (X,A) is a strong G-NDR pair if (h,u) is a G-NDR representation of (X,A) that is also a strong NDR pair representation for (X,A).

Given an NDR pair (X,A) represented by (h,u), an integer $n \geq 0$ and $1 \leq r \leq n$, denote by F_rX^n the subspace of X^n consisting of n-tuples $(x_1,...,x_n)$ where at least r of the x_j's are in A. In [3], it is proved that (X^n, F_rX^n) is an NDR-pair. This naturally extends to the equivariant situation as it is shown next.

Lemma 4.4. *Suppose that (X,A) is a G-NDR pair. Then the pair (X^n, F_rX^n) is a $G\times\Sigma_n$-equivariant NDR pair represented by $(h_n, u_{r,n})$, where*

$$u_{r,n}(x_1,...,x_n)=\frac{1}{r}\min_{1\le i_1<\cdots<i_r\le n}(u(x_{i_1})+\cdots+u(x_{i_r}))$$

and

$$h_n((x_1,...,x_n),t)=(h(x_1,t_1),...,h(x_n,t_n)),$$

with

$$t_i=\begin{cases} t\min_{i\neq n}(u(x_m)/u(x_i)) & \text{if some } u(x_m)<u(x_i),\ m\neq i,\\ t & \text{if all } u(x_m)\ge u(x_i).\end{cases}$$

Proof: By [3, Lemma 7.2] $(h_n, u_{r,n})$ is a Σ_n-NDR-pair representation for (X^n, F_rX^n). Thus it is only necessary to see that $(h_n, u_{r,n})$ respects the G-action, where G acts diagonally on X^n. To see this note that F_rX^n is G-invariant as A is G-invariant, also for any $g\in G$

$$\begin{aligned} u_{r,n}(g\cdot x_1,...,g\cdot x_n) &= \frac{1}{r}\min_{1\le i_1<\cdots<i_r\le n}(u(g\cdot x_{i_1})+\cdots+u(g\cdot x_{i_r}))\\ &= \frac{1}{r}\min_{1\le i_1<\cdots<i_r\le n}(u(x_{i_1})+\cdots+u(x_{i_r}))\end{aligned}$$

as u is G-invariant. On the other hand, for any $t\in[0,1]$ and t_i as above for $0\le i\le n$,

$$\begin{aligned} h_n((g\cdot x_1,...,g\cdot x_n),t) &= (h(g\cdot x_1,t_1),...,h(g\cdot x_n,t_n))\\ &= (g\cdot h(x_1,t_1),...,g\cdot h(x_n,t_n))\\ &= g\cdot h_n((x_1,...,x_n),t).\end{aligned}$$

Therefore $(h_n,u_{r,n})$ is a $G\times\Sigma_n$-NDR pair representation for (X^n,F_rX^n). □

The case $X=G$ and $A=\{1_G\}$, where G is a Lie group is of particular interest. In this case $F_rG^n\subset G^n$ is the subspace of n-tuples $(x_1,...,x_n)$ with at least r of the x_i's equal to 1_G. Let $S_n^r(G,K):=B_n(G,K)\cap F_rG^n$, thus $S_n^r(G,K)\subset G^n$ is the subspace of K-almost commuting n-tuples with at least r components equal to 1_G. Note that G acts by conjugation on itself and that each $S_n^r(G,K)$ is invariant under this action. Denote by $\bar{S}_n^r(G,K)$ the orbit space $S_n^r(G,K)/G$. For any Lie group G, the pair $(G,\{1_G\})$ is an NDR pair and thus by the previous lemma $(G^n, S_n^r(G))$ is an NDR pair. The following proposition provides a criterion that can be used to show that each pair $(S_n^{r-1}(G,K), S_n^r(G,K))$ is a strong G-NDR pair when G is a compact Lie group.

Proposition 4.5. *Let G be a Lie group and $K\subset Z(G)$ a closed subgroup. Suppose (h,u) is a representation of $(G,\{1_G\})$ as a strong NDR pair with the following additional properties: for each $g\in G-\{1_G\}$ and any $0\le t<1$*

1. *$Z(g)=Z(h(g,t))$ and*
2. *$\{x\in G\mid [x,g]\in K\}=\{x\in G\mid [x,h(g,t)]\in K\}$.*

Then for each $0\le r\le n$ the representation $(h_n,u_{r,n})$ as above restricts to a representation of $(S_n^{r-1}(G,K), S_n^r(G,K))$ as a Σ_n-equivariant strong NDR pair. Moreover, if the representation (h,u) is G-equivariant, G acting on itself by conjugation, then the pair $(S_n^{r-1}(G,K), S_n^r(G,K))$ is a strong G-NDR pair.

Proof: Suppose first that $(x_1, ..., x_n) \in B_n(G, K)$; that is, $[x_i, x_j] \in K$ for all i and j. Thus if t_i and t_j are as explained before for each i and j then

$$x_i \in \{x \in G \mid [x, x_j] \in K\} = \{x \in G \mid [x, h(x_j, t_j)] \in K\},$$

where the last equality holds by assumption (2). Thus

$$[h(x_j, t_j), x_i] = [x_i, h(x_j, t_j)]^{-1} \in K$$

and therefore

$$h(x_j, t_j) \in \{x \in G \mid [x, x_i] \in K\} = \{x \in G \mid [x, h(x_i, t_i)] \in K\}.$$

From here it can be concluded that $[h(x_i, t_i), h(x_j, t_j)] \in K$ for all i and j, hence

$$h_n((x_1, ..., x_n), t) = (h(x_1, t_1), ..., h(x_n, t_n)) \in B_n(G, K).$$

Therefore $h_n : B_n(G, K) \times [0, 1] \to B_n(G, K)$ is well defined. Moreover each $S_n^r(G, K)$ is invariant under each $h_n(\cdot, t)$ as if $(x_1, ..., x_n) \in S_n^r(G, K)$, then at least r of the x_i's are equal to 1_G. Since $h(1_G, t) = 1_G$ for all t, it follows that $h_n((x_1, ..., x_n), t)$ has at least r components equal to 1_G. Let $(\tilde{h}_n, \tilde{u}_{r,n})$ be the restriction of $(h_n, u_{r,n})$ to $S_n^{r-1}(G, K)$. Then $(\tilde{h}_n, \tilde{u}_{r,n})$ represents the pair $(S_n^{r-1}(G, K), S_n^r(G, K))$ as a strong NDR pair. For this to be true conditions (1)-(4) of the definition of an NDR pair and the extra condition of a strong NDR pair need to be verified. Indeed,

(1) Note that $\tilde{u}_{r,n}^{-1}(0) = S_n^{r-1}(G, K) \cap u_{r,n}^{-1}(0)$, but $u_{r,n}^{-1}(0) = F_rG^n$ and therefore $\tilde{u}_{r,n}^{-1}(0) = S_n^{r-1}(G, K) \cap F_rG^n = S_n^r(G, K)$.
(2) Since $h_n(x, 0) = x$ for all $x \in G^n$, then trivially $\tilde{h}_n(x, 0) = x$ for all $x \in S_n^{r-1}(G, K)$.
(3) $h_n(a, t) = a$ for all $a \in F_rG^n$, since $S_n^r(G, K) \subset F_rG^n$ it follows that $\tilde{h}_n(a, t) = a$ for all $a \in S_n^r(G, K)$.
(4) If $w \in \tilde{u}_{r,n}^{-1}([0, 1)) = u_{r,n}^{-1}([0, 1)) \cap S_n^{r-1}(G, K)$, then $w \in u_{r,n}^{-1}([0, 1)) = F_rG^n$ and thus $w \in F_rG^n \cap S_n^{r-1}(G, K) = S_n^r(G, K)$ and the converse is also true. Thus, $\tilde{u}_{r,n}^{-1}([0, 1)) = S_n^r(G, K)$.

From the above $(\tilde{h}_n, \tilde{u}_{r,n})$ is an NDR representation of $(S_n^{r-1}(G, K), S_n^r(G, K))$. Suppose now that $(x_1, ..., x_n) \in S_n^{r-1}(G, K)$ is such that $u(x_1, ..., x_n) < 1$. By definition

$$u_{r,n}(x_1, ..., x_n) = \frac{1}{r} \min_{1 \leq i_1 < \cdots < i_r \leq n} \left(u(x_{i_1}) + \cdots + u(x_{i_r})\right).$$

It follows that for some r-tuple $1 \leq j_1 < \cdots < j_r \leq n$

$$\frac{1}{r} \left(u(x_{j_1}) + \cdots + u(x_{j_r})\right) < 1.$$

Since $0 \leq u(x) \leq 1$ for all $x \in X$, this happens if and only if $u(x_{j_s}) < 1$ for some $1 \leq s \leq r$. Since (h, u) is a strong NDR pair representation of $(G, \{1_G\})$ then $u(h(x_{j_s}, t)) < 1$ for all $t \in [0, 1]$. On the other hand, by definition

$$u(h((x_1, ..., x_n), t)) = \frac{1}{r} \min_{1 \leq i_1 < \cdots < i_r \leq n} \left(u(h(x_{i_1}, t_{i_1})) + \cdots + u(h(x_{i_r}, t_{i_r}))\right).$$

Here $t_1, ..., t_n$ are as explained before. For the r-tuple $1 \leq j_1 < \cdots < j_r \leq n$

$$\frac{1}{r}(u(h(x_{j_1}, t_{j_1})) + \cdots + u(h(x_{j_r}, t_{j_r}))) < 1$$

as $u(h(x_{j_s}, t_{j_s})) < 1$. Thus in particular

$$u(h((x_1, ..., x_n), t)) < 1.$$

To finish, note that $(S_n^{r-1}(G,K), S_n^r(G,K))$ is G-pair and if (h,u) is a G-NDR pair representation of $(G, \{1_G\})$, then as noted above each h_n is G-equivariant and $u_{n,r}$ is G-invariant. This shows that $(\tilde{h}_n, \tilde{u}_{r,n})$ is in this case a G-NDR representation of $(S_n^{r-1}(G,K), S_n^r(G,K))$. □

Next it is shown that if G is any compact Lie group and $K \subset Z(G)$ is a closed subgroup then such a G-equivariant NDR representation (h,u) satisfying the additional properties required in the last proposition can always be found. Recall that given any Lie group G, if $\mathfrak{g}$ denotes the Lie algebra of G, then the adjoint representation gives a homomorphism

$$Ad : G \to Aut(\mathfrak{g}).$$

For each $x \in G$, Ad_x is the derivative of the the map

$$\begin{aligned} \tau_x : G &\to G \\ g &\mapsto xgx^{-1}. \end{aligned}$$

The Lie group G can be seen as a Riemannian manifold. In particular $\mathfrak{g}$ is a normed space with norm $\|\cdot\|$. In this way Ad can be seen as a continuous family $\{Ad_x\}_{x\in G}$ of bounded operators on the normed space $\mathfrak{g}$ that is parametrized by G. Since G is assumed to be a compact Lie group, there exists a Haar measure μ on G. As usual, the average of the norm $\|\cdot\|$ can be used to obtain a new norm $\|\cdot\|_{\mathfrak{g}}$ in $\mathfrak{g}$ with the further property that each Ad_x is an isometry for all $x \in G$. Indeed, define

$$\|v\|_{\mathfrak{g}} = \frac{1}{\mu(G)} \int_{x\in G} \|Ad_x v\| \, d\mu(x),$$

then $\|\cdot\|_{\mathfrak{g}}$ is a norm on $\mathfrak{g}$ that satisfies the required additional property. From now on such a norm on $\mathfrak{g}$ is fixed. Note that in the particular case when G is semisimple and compact, the norm $\|\cdot\|_{\mathfrak{g}}$ can be taken to be the negative of the Killing form which is positive definite and non-degenerate.

On the other hand, for any Lie group the exponential map is a local homeomorphism, in particular there exists an $\epsilon > 0$ such that the restriction of the exponential map to $\bar{B}_\epsilon(0)$

$$\exp : \bar{B}_\epsilon(0) \to G$$

is a homeomorphism onto its image. Moreover, since $K \subset G$ is a closed subgroup, ϵ can be chosen small enough so that $\exp(B_\epsilon(0) \cap \mathfrak{k}) = \exp(B_\epsilon(0)) \cap K$, where $\mathfrak{k}$ is the Lie algebra of K. This is used to prove the following lemma.

Lemma 4.6. *Let G be a compact Lie group and $K \subset Z(G)$ a closed subgroup. Choose $\epsilon > 0$ as above and let $y \in \bar{B}_{\epsilon/2}(0) - \{0\}$. Then for any $0 < t \leq 1$*

1. *$Z_G(\exp(y)) = Z_G(\exp(ty))$ and*
2. *$\{x \in G \mid [x, \exp(y)] \in K\} = \{x \in G \mid [x, \exp(ty)] \in K\}$.*

Proof: Part (1) was proved in [3, Lemma 8.1]. To prove (2), suppose that $c \in K \subset Z(G)$. Take $x \in G$ with $[x, \exp(y)] = c$; that is, $x(\exp(y))x^{-1}(\exp(-y)) = c$. Since

$$x(\exp(y))x^{-1} = \exp(Ad_x y),$$

it follows that $\exp(Ad_x y)\exp(-y) = c \in Z(G)$. In particular $\exp(Ad_x y)$ and $\exp(-y)$ commute. Indeed, if $z, w \in G$ are such that $zw = c$ for c central, then $w = z^{-1}c = cz^{-1}$ which implies that $zw = c = wz$. By part (1) it follows that $\exp(Ad_x ty)$ and $\exp(-ty)$ commute for all $0 \leq t \leq 1$. By [16, Lemma II 2.1]

$$1 = \exp(ty)\exp(-Ad_x ty)\exp(-ty)\exp(Ad_x ty) = \exp(-t^2[y, Ad_x y] + O(t^3)),$$

where $(1/t^3)O(t^3)$ is bounded and analytic for t sufficiently small. This shows that $[y, Ad_x y] = 0$. Therefore

$$c = \exp(Ad_x y)\exp(-y) = \exp(Ad_x y - y).$$

Note that $(Ad_x y - y) \in \mathfrak{g}$ is such that

$$\|Ad_x y - y\|_{\mathfrak{g}} \leq \|Ad_x - I\|_{\mathfrak{g}} \cdot \|y\|_{\mathfrak{g}} \leq (\|Ad_x\|_{\mathfrak{g}} + \|I\|_{\mathfrak{g}})\frac{\epsilon}{2} \leq \epsilon.$$

The last inequality follows from the fact that $\|Ad_x\|_{\mathfrak{g}} = 1$ as Ad_x is an isometry with respect to the norm $\|\cdot\|_{\mathfrak{g}}$. This means that $(Ad_x y - y) \in \bar{B}_\epsilon(0)$. Since $c \in K$ and $c = \exp(Ad_x y - y)$ with $(Ad_x y - y) \in \bar{B}_\epsilon(0)$, then as exp is injective on $\bar{B}_\epsilon(0)$ it follows that $k = (Ad_x y - y) \in \mathfrak{k}$ where $\mathfrak{k}$ is the Lie algebra of K. In particular,

$$\exp(t(Ad_x y - y)) = \exp(tk) \in K \text{ for } t \in \mathbb{R}.$$

Unraveling the definitions, this means that

$$[x, \exp(ty)] \in K \text{ for all } t \in \mathbb{R}.$$

Since $c \in K$ was arbitrary this shows that

$$\{x \in G \mid [x, \exp(y)] \in K\} \subset \{x \in G \mid [x, \exp(ty)] \in K\}.$$

If $t \neq 0$, then the same argument with $s = 1/t$ shows the other inclusion. Hence

$$\{x \in G \mid [x, \exp(y)] \in K\} = \{x \in G \mid [x, \exp(ty)] \in K\}.$$

□

To finish this section, it is shown that if G is any compact Lie group and $K \subset Z(G)$ is a closed subgroup then the previous lemma can be used to construct a strong G-equivariant NDR representation (h, u) for the pair $(G, \{1_G\})$ that satisfies the additional properties required in Proposition 4.5. Take $\epsilon > 0$ as before. Define a function $u : G \to [0, 1]$ as follows

$$u(g) = \begin{cases} 2\|y\|_{\mathfrak{g}}/\epsilon & \text{if } g = \exp(y) \text{ for } g \in \exp(\bar{B}_{\epsilon/2}(0)), \\ 1 & \text{if } g \in G - \exp(B_{\epsilon/2}(0)). \end{cases}$$

Note that u is continuous with $u^{-1}(0) = \{1_G\}$ and $u^{-1}([0, 1)) = \exp(\bar{B}_{\epsilon/2}(0))$ and that u is invariant under the conjugation action of G. To see this, suppose $x \in G$, then since Ad_x is an isometry for the norm $\|\cdot\|_{\mathfrak{g}}$

$$u(xgx^{-1}) = \begin{cases} 2\|Ad_x(y)\|_{\mathfrak{g}}/\epsilon = 2\|y\|_{\mathfrak{g}}/\epsilon & \text{if } g = \exp(y) \text{ for } g \in \exp(B_{\epsilon/2}(0)), \\ 1 & \text{if } g \in G - \exp(B_{\epsilon/2}(0)). \end{cases}$$

On the other hand, take $s : G \to [0, 1]$ any G-invariant continuous function satisfying the following properties

$$s(g) = \begin{cases} 1 & \text{if } g = \exp(y) \text{ for } g \in \exp(\bar{B}_{\epsilon/2}(0)), \\ 0 & \text{if } g \in G - \exp(B_\epsilon(0)). \end{cases}$$

Clearly such a bump function s always exists and can be constructed using partitions of unity or by hand using the exponential map as done in [3]. The map s can be made to be G-invariant by the usual averaging trick using a Haar measure as before. Finally, define a homotopy

$$h : G \times [0,1] \to G$$

by

$$h(g,t) = \begin{cases} \exp((1-t)y) & \text{if } g = \exp(y) \text{ for } y \in \bar{B}_{\epsilon/2}(0), \\ \exp((1-s(g)t)y) & \text{if } g = \exp(y) \text{ for } y \in \bar{B}_{\epsilon}(0) - B_{\epsilon/2}(0), \text{ and} \\ g & \text{if } g \in G - \exp(B_{\epsilon}(0)). \end{cases}$$

The map h is G-equivariant. To see this, take $x \in G$, then as Ad_x is an isometry with respect to the metric $\|\cdot\|_{\mathfrak{g}}$ the following are true

$$h(xgx^{-1},t) = \exp((1-t)Ad_x y) = x\exp((1-t)y)x^{-1} = xh(g,t)x^{-1}$$

if $g = \exp(y)$ for $y \in \bar{B}_{\epsilon/2}(0)$,

$$h(xgx^{-1},t) = \exp((1-s(xgx^{-1})t)Ad_x y) = x\exp((1-s(x)t)y)x^{-1} = xh(g,t)x^{-1}$$

if $g = \exp(y)$ for $y \in \bar{B}_{\epsilon}(0) - B_{\epsilon/2}(0)$, and

$$h(xgx^{-1},t) = xgx^{-1} = xh(g,t)x^{-1}$$

if $g \in G - \exp(B_{\epsilon}(0))$. From here it follows that h is G-equivariant, where G is acting on itself by conjugation. The previous remarks can be used to prove the following proposition.

Proposition 4.7. *Let G be a compact Lie group and $K \subset Z(G)$ a closed subgroup. Then the pair (h,u) as defined above is a strong G-NDR pair representation of $(G,\{1_G\})$ that satisfies the following additional properties: for each $g \in G - \{1_G\}$ and any $0 \le t < 1$*

(1) *$Z(g) = Z(h(g,t))$ and*
(2) *$\{x \in G \mid [x,g] \in K\} = \{x \in G \mid [x,h(g,t)] \in K\}$.*

Proof: It can be seen directly, as it was done in [3, Proposition 8.2] that (h,u) is an NDR pair representation for $(G,\{1_G\})$. By the previous remarks, h is G-equivariant and u is G-invariant, thus (h,u) is a G-NDR pair representation for $(G,\{1_G\})$. Moreover, (h,u) is a strong G-NDR pair representation for $(G,\{1_G\})$. To see this, suppose that $g \in G$ is such that $u(g) < 1$. It follows that $g = \exp(y)$ for some $y \in B_{\epsilon/2}(0)$. Therefore $h(g,t) = \exp((1-t)y)$ and

$$u(h(g,t)) = u(\exp((1-t)y)) = 2\,|1-t|\,\|y\|_{\mathfrak{g}}/\epsilon \le 2\,\|y\|_{\mathfrak{g}}/\epsilon < 1,$$

for all $t \in [0,1]$. Conditions (1) and (2) are now verified. To do so, note that for any $t \in [0,1]$ and any $g \in G$, $h(g,t)$ is either 1_G, g or $\exp(ky)$ for $0 < k \le 1$ and $y \in \bar{B}_{\epsilon}(0)$ such that $g = \exp(y)$. Moreover, if $g \ne 1$ and $t \ne 1$, then $h(g,t) \ne 1$ and thus $h(g,t)$ is either g or $\exp(ky)$ for $0 < k \le 1$. The proposition follows then by Lemma 4.6. □

The following theorem which is the goal of this section is an immediate corollary of Propositions 4.5 and 4.7.

Theorem 4.8. *If G is a compact Lie group and $K \subset Z(G)$ is a closed subgroup then, for each $n \ge 0$ and each $0 \le r \le n$ the pair $(S_n^{r-1}(G,K), S_n^r(G,K))$ is a strong G-NDR pair. In particular, $(\bar{S}_n^{r-1}(G,K), \bar{S}_n^r(G,K))$ and $(S_n^{r-1}(G,K)^H, S_n^r(G,K)^H)$ are strong NDR pairs for every subgroup $H \subset G$.*

Remark. The previous theorem can also be proved as follows. Let G be a compact Lie group and $K \subset Z(G)$ a closed subgroup. Then using the methods in [21] it can be proved that the spaces of the form $S^r_n(G, K)$ are semi-algebraic sets and that these have the homotopy type of a G-CW complex (see [21] for definitions). Moreover, $S^r_n(G, K)$ can be seen as a G-subcomplex of $S^{r-1}_n(G, K)$ for all $1 \leq r \leq n$. The authors would like to thank one of the referees for pointing out this alternative approach. The arguments provided here can be extended in some situations to groups which are not necessarily compact (see [3] for example).

5. Stable decompositions

In [3], it was proved that if G is a closed subgroup of $GL(n, \mathbb{C})$, in particular if G is a compact Lie group, there is a natural homotopy equivalence

$$\Sigma(\mathrm{Hom}(\mathbb{Z}^n, G)) \simeq \bigvee_{1\leq r\leq n} \Sigma\left(\bigvee^{\binom{n}{r}} \mathrm{Hom}(\mathbb{Z}^r, G)/S_r(G)\right).$$

Here $S_r(G) \subset \mathrm{Hom}(\mathbb{Z}^r, G)$ is the subspace of r-tuples $(x_1, ..., x_r) \in \mathrm{Hom}(\mathbb{Z}^r, G)$ for which at least one of the x_i's equals 1_G. In this section it is proved that a similar stable splitting holds for the spaces $B_n(G, K)$, when G is a compact Lie group and $K \subset Z(G)$ is a closed subgroup.

Let X_* be a simplicial space with face and degeneracy maps $\partial_i : X_n \to X_{n-1}$ and $s_j : X_n \to X_{n+1}$ respectively. The different degeneracy maps can be used to obtain a filtration of every space X_n as follows. For each $0 \leq r \leq n$, denote

$$S^r(X_n) = \bigcup_{0\leq j_r<\cdots<j_1\leq n} s_{j_1}s_{j_2}\cdots s_{j_r}(X_{n-r}).$$

Also define $S^{n+1}(X_n)$ to be the empty set. These spaces form a filtration of X_n

$$\emptyset = S^{n+1}(X_n) \subset S^n(X_n) \subset \cdots \subset S^1(X_n) \subset S^0(X_n) = X_n.$$

Definition 5.1. A simplicial space X_* is said to be simplicially NDR if for every $n \geq 0$ and every $0 \leq r \leq n$ the pair $(S^r(X_n), S^{r+1}(X_n))$ is an NDR pair.

Under some circumstances the previous filtration splits stably. For example, the following theorem proved in [2] establishes that if a simplicial space X_* is simplicially NDR, then previous filtration of X_n splits after one suspension.

Theorem 5.2. *Consider a simplicial space X_* that is simplicially NDR. Then for every $n \geq 0$ there is a natural homotopy equivalence*

$$\Theta(n) : \Sigma(X_n) \to \bigvee_{0\leq r\leq n} \Sigma(S^r(X_n)/S^{r+1}(X_n)).$$

In general for a simplicial space X_* there is a natural filtration

$$F_0 |X_*| \subset F_1 |X_*| \subset \cdots \subset F_n |X_*| \subset \cdots \subset |X_*|$$

of $|X_*|$, the geometric realization of X_*. The space $F_j |X_*|$ is precisely the image in $|X_*|$ of $\sqcup_{0\leq k\leq j} X^k \times \Delta^k$. This filtration behaves nicely if X_* satisfies certain conditions. One such condition is the following definition that was given by May in [19].

Definition 5.3. A simplicial space X_* is proper if each pair $(X_n, S^1(X_n))$ is a strong NDR pair.

When a simplicial space X_* is simplicially NDR and proper, then the stable homotopy type of the factors $S^r(X_n)/S^{r+1}(X_n)$ that appear in the previous theorem can be identified in terms of the filtration $F_n|X_*|$ of $|X_*|$. More precisely, there is the following theorem also proved in [2].

Theorem 5.4. *Let X_* be a simplicial space that is proper and simplicially NDR. Then there are homotopy equivalences*

$$K(n,r) : \Sigma^{n+1}(S^r(X_n)/S^{r+1}(X_n)) \to \bigvee_{J_r} \Sigma^{r+1}(F_{n-r}|X_*|/F_{n-r-1}|X_*|).$$

Thus by Theorem 5.2 there are natural homotopy equivalences

$$\Theta'(n) : \Sigma^{n+1}(X_n) \to \bigvee_{0\leq r\leq n} \bigvee_{J_r} \Sigma^{r+1}(F_{n-r}|X_*|/F_{n-r-1}|X_*|),$$

where J_r runs over all possible sequences of the form $0 \leq j_r < \cdots < j_1 \leq n$.

As pointed out before, the collection $B_*(G,K)$ forms a simplicial space for a general Lie group and a closed central subgroup K. By definition for $0 \leq r \leq n$ the space $S^r(B_n(G,K))$ in the filtration of $B_n(G,K)$ given by the degeneracy maps is precisely $S_n^r(G,K)$. In Theorem 4.8, it was proved that if G is a compact Lie group and $K \subset Z(G)$ is a closed subgroup then for every $n \geq 0$ and every $0 \leq r \leq n$ the pair $(S_n^r(G,K), S_n^{r+1}(G,K))$ is a strong NDR pair. Thus the following is obtained as a corollary of Theorem 4.8:

Corollary 5.5. *Let G be a compact Lie group and $K \subset Z(G)$ a closed subgroup. Then the simplicial space $B_*(G,K)$ is proper and simplicially NDR.*

A stable splitting for $B_n(G,K)$ is obtained as a direct consequence of the previous corollary. Moreover, the different quotients

$$S^r(B_n(G,K))/S^{r+1}(B_n(G,K))$$

can be identified in the same way as in [3, Section 6]. To be more precise, for $0 \leq r \leq n$, let $J_{n,r}$ denote the set of all sequences of the form

$$1 \leq m_1 < \cdots < m_{n-r} \leq n.$$

Note that $J_{n,r}$ contains precisely $\binom{n}{n-r} = \binom{n}{r}$ elements. Given such a sequence, there is an associated projection

$$\begin{aligned} P_{m_1,\ldots,m_{n-r}} : B_n(G,K) &\to B_{n-r}(G,K) \\ (x_1,\ldots,x_n) &\mapsto (x_{m_1},\ldots,x_{m_{n-r}}). \end{aligned}$$

These projections are G-equivariant, with G acting by conjugation and can be assembled to obtain a G-map

$$\begin{aligned} \eta_n : B_n(G,K) &\to \prod_{J_{n,r}} B_{n-r}(G,K)/S^1_{n-r}(G,K) \\ (x_1,\ldots,x_n) &\mapsto \{\bar{P}_{m_1,\ldots,m_{n-r}}((x_1,\ldots,x_n))\}_{(m_1,\ldots,m_{n-r})\in J_{n,r}}. \end{aligned}$$

The map η_n sends the G-invariant space $S^r_n(G,K)$ onto $\bigvee_{J_{n,r}} B_{n-r}(G,K)/S^1_{n-r}(G,K)$ and $S^{r+1}_n(G,K)$ is mapped onto the base point. Therefore η_n induces a G-equivariant continuous map

$$\begin{aligned} S^r(B_n(G,K))/S^{r+1}(B_n(G,K)) &\to \bigvee_{J_{n,r}} B_{n-r}(G,K)/S^1_{n-r}(G,K) \\ &= \bigvee^{\binom{n}{r}} B_{n-r}(G,K)/S^1_{n-r}(G,K) \end{aligned}$$

and this map is easily shown to be a G-equivariant homeomorphism. This together with Theorems 5.2 and 5.4 can be used to prove the following theorem.

Theorem 5.6. *Suppose that G is a compact Lie group and that $K \subset Z(G)$ is a closed subgroup. Then for each $n \geq 1$ there is a natural G-equivariant homotopy equivalence*

$$\Theta(n) : \Sigma(B_n(G,K)) \simeq \bigvee_{1\leq r\leq n} \Sigma\left(\bigvee^{\binom{n}{r}} B_r(G,K)/S^1_r(G,K)\right).$$

In particular there is natural homotopy equivalence

$$\bar{\Theta}(n) : \Sigma(\bar{B}_n(G,K)) \simeq \bigvee_{1\leq r\leq n} \Sigma\left(\bigvee^{\binom{n}{r}} \bar{B}_r(G,K)/\bar{S}^1{}_r(G,K)\right).$$

Proof: By Theorems 5.2 and 5.5 and the previous remark it follows that $\Theta(n)$ is a homotopy equivalence. To see that this is in fact a G-equivariant homotopy equivalence, note that if $g \in G$, then the map conjugation by g defines a map of simplicial spaces

$$\tau_g : B_*(G,K) \to B_*(G,K).$$

By naturality it follows that $\Theta(n)$ is a G-equivariant map, with G acting by conjugation. On the other hand, if $H \subset G$ is a subgroup then $B_n(G,K)^H$ forms a simplicial space. By Theorem 4.8 and Lemma 4.2 the pair $(S^{r-1}_n(G,K)^H, S^r_n(G,K)^H)$ is a strong NDR pair. This means that the simplicial space $B_*(G,K)^H$ is simplicially NDR and Theorem 5.2 applied to this simplicial space provides a homotopy equivalence

$$\Theta(n,H) : \Sigma(B_n(G,K)^H) \simeq \bigvee_{1\leq r\leq n} \Sigma\left(\bigvee^{\binom{n}{r}} B_r(G,K)^H/S^1_r(G,K)^H\right).$$

The map $\Theta(n,H)$ agrees by naturality with the fixed point set map $\Theta(n)^H$. Therefore $\Theta(n)$ is a G-equivariant map such that for every subgroup $H \subset G$, the fixed point map $\Theta(n)^H$ is a homotopy equivalence. In particular $\Theta(n)$ is a G-equivariant weak homotopy equivalence. By Proposition 5.7 below the G-space $B_n(G,K)$ has the homotopy type of a G-CW complex (similarly each $B_r(G,K)/S^1_r(G,K)$ has the homotopy type of a G-CW complex for $1 \leq r \leq n$) and thus the theorem follows by the equivariant Whitehead Theorem. $\square$

Remark. The naturality of the map $\Theta(n)$ in the previous theorem is with respect to morphisms between pairs (G, K) and (G', K'); that is, $\Theta(n)$ is natural with respect to homomorphisms $f : G \to G'$ of Lie groups such that $f(K) \subset K'$.

Proposition 5.7. *Let G be a compact Lie group and $K \subset Z(G)$ a closed subgroup. Then for every $n \geq 1$ the space $B_n(G, K)$ has the homotopy type of a G-CW complex.*

Proof: The case $K = \{1_G\}$ is considered first, this corresponds to the space of commuting n-tuples in G. Since G is a compact Lie group it is well known that G carries the structure of a real algebraic group. This is why $B_n(G, \{1_G\}) = \mathrm{Hom}(\mathbb{Z}^n G) \subset G^n$ is a compact real algebraic G-variety, where the action of G is given by conjugation. By [21, Theorem 1.3] it follows that $\mathrm{Hom}(\mathbb{Z}^n G)$ has the homotopy type of a G-CW complex. Suppose now that $K \subset Z(G)$ is any closed subgroup. By the previous argument, the space $\mathrm{Hom}(\mathbb{Z}^n, G/K)$ has the homotopy type of a G/K-CW complex. The natural map $p : G \to G/K$ induces a G action on $\mathrm{Hom}(\mathbb{Z}^n, G/K)$ and this way $\mathrm{Hom}(\mathbb{Z}^n, G/K)$ has the homotopy type of a G-CW complex. By Lemma 2.3 the natural map $\phi_n : B_n(G, K) \to \mathrm{Hom}(\mathbb{Z}^n, G/K)$ is a G-equivariant locally trivial principal K^n-bundle. The group G acts trivially on K^n as K is central and this is enough to conclude that $B_n(G, K)$ has the homotopy type of a G-CW complex. To see this, note that $B_n(G, K)$ is a separable metric space and thus by [20, Theorem 14.2] it is enough to show that $B_n(G, K)$ has the homotopy type of a G-ANR. By [20, Theorem 9.5] this is equivalent to showing that every point $\underline{x} \in B_n(G, K)$ has a $G_{\underline{x}}$-neighborhood which is a $G_{\underline{x}}$-ANR but the latter follows easily from the local triviality of the principal K^n-bundle $B_n(G, K) \to \mathrm{Hom}(\mathbb{Z}^n, G/K)$ and the fact that G acts trivially on K^n. □

Using [19, Lemma 11.3] and the fact that $B_*(G, K)$ is a proper simplicial space when G is a compact Lie group, the stable factors appearing in the previous Theorem 5.6 can be described in terms of the natural filtration of $B(G, K)$, the geometric realization of $B_*(G, K)$.

Proposition 5.8. *Let G be a compact Lie group and $K \subset Z(G)$ a closed subgroup. Then there is a G-equivariant homeomorphism*

$$\Sigma^n B_n(G, K)/S^1_n(G, K) \cong F_n B(G, K)/F_{n-1} B(G, K),$$

where

$$F_0 B(G, K) \subset F_1 B(G, K) \subset \cdots \subset F_n B(G, K) \subset \cdots \subset B(G, K)$$

is the natural filtration of $B(G, K)$.

6. Representation spaces and symmetric products

In this section the representation spaces $\mathrm{Rep}(\mathbb{Z}^n, G)$ are studied for compact Lie groups. In particular the stable splitting of Theorem 5.6 is completely determined for these spaces when G is such that $\mathrm{Rep}(\mathbb{Z}^n, G)$ is path-connected for all $n \geq 1$. Also, a connection between representation spaces and symmetric products is explored from which interesting consequences can be derived.

To begin, take G to be a compact Lie group. If $K = \{1_G\}$ is the trivial group then by definition $B_n(G, \{1_G\}) = \mathrm{Hom}(\mathbb{Z}^n, G)$ and Theorem 5.6 provides a stable splitting for $\mathrm{Hom}(\mathbb{Z}^n, G)$. This splitting agrees with [3, Theorem 1.6] with the additional fact that the splitting map $\Theta(n)$

is a G-equivariant homotopy equivalence. In particular, after passing to orbit spaces, $\bar{\Theta}(n)$ defines a natural homotopy equivalence

$$\bar{\Theta}(n) : \Sigma\mathrm{Rep}(\mathbb{Z}^n, G) \simeq \bigvee_{1\leq r\leq n} \Sigma\left(\bigvee^{\binom{n}{r}} \mathrm{Rep}(\mathbb{Z}^n, G)/\bar{S}^1_r(G)\right).$$

The following theorem identifies the stable pieces $\mathrm{Rep}(\mathbb{Z}^n, G)/\bar{S}^1_r(G)$ under the assumption that $\mathrm{Rep}(\mathbb{Z}^r, G)$ is path-connected for all $r \geq 1$.

Theorem 6.1. *Let G be a compact, connected Lie group such that $Rep(\mathbb{Z}^r, G)$ is path-connected for $1 \leq r \leq n$. Let T be a maximal torus of G and W the Weyl group associated to T. Then there is a homeomorphism*

$$Rep(\mathbb{Z}^n, G) \cong T^n/W,$$

where W acts diagonally. Moreover, the map $\bar{\Theta}(n)$ defines a homotopy equivalence

$$\Sigma Rep(\mathbb{Z}^n, G) \simeq \bigvee_{1\leq r\leq n} \Sigma(\bigvee^{\binom{n}{r}} T^{\wedge r}/W),$$

where $T^{\wedge r}$ is the smash product of r copies of T.

Proof: Consider the continuous map

$$\begin{aligned}\varphi_n : G \times T^n &\to \mathrm{Hom}(\mathbb{Z}^n, G)\\ (g, t_1, ..., t_n) &\mapsto (gt_1g^{-1}, ..., gt_ng^{-1}).\end{aligned}$$

Under the given hypothesis, [8, Lemma 4.2] shows that every commuting n-tuple in G lies in a maximal torus of G. Since any two maximal tori in G are conjugated then φ_n is surjective. Moreover, φ_n is invariant under the action of $N(T)$ and therefore it induces a continuous map

$$\bar{\varphi}_n : G \times_{N(T)} T^n \to \mathrm{Hom}(\mathbb{Z}^n, G).$$

This map is G-equivariant, where G acts by left multiplication on the G factor of $G \times_{N(T)} T^n$ and by conjugation on $\mathrm{Hom}(\mathbb{Z}^n, G)$. The induced map on the level of orbit spaces is a homeomorphism

$$\psi_n : T^n/W \cong \mathrm{Rep}(\mathbb{Z}^n, G).$$

Under this homeomorphism, $\bar{S}^1_r(G)$ corresponds to the subspace of T^n/W consisting of equivalence classes of the form $[t_1, ..., t_n]$ for which at least one $t_i = 1$. Therefore ψ_n defines a homeomorphism

$$T^{\wedge n}/W \cong \mathrm{Rep}(\mathbb{Z}^n, G)/\bar{S}^1_r(G).$$

□

Remark. By [3, Proposition 2.3] if a Lie group G is such that every abelian subgroup of G is contained in a path-connected abelian subgroup, then the spaces $\mathrm{Hom}(\mathbb{Z}^n, G)$ and $\mathrm{Rep}(\mathbb{Z}^n, G)$ are path-connected for every n. This is the case for $G = U(m)$, $SU(m)$ and $Sp(m)$.

Example 6.2. Let $G = SU(m)$ with $m \geq 1$. The space $\mathrm{Rep}(\mathbb{Z}^n, SU(m))$ is path-connected by the previous remark. In this case a maximal torus T can be taken to be the subspace of diagonal matrices with entries in $\mathbb{S}^1$ and determinant one, thus $T \cong (\mathbb{S}^1)^{m-1}$. The Weyl group W is the symmetric group Σ_m acting on T by permuting the diagonal entries of a matrix in T. According to the previous theorem there is a homotopy equivalence

$$\Sigma\mathrm{Rep}(\mathbb{Z}^n, SU(m)) \simeq \bigvee_{1\leq r\leq n} \Sigma\left(\bigvee^{\binom{n}{r}}(\underbrace{(\mathbb{S}^1)^{m-1}\wedge\cdots\wedge(\mathbb{S}^1)^{m-1}}_{r-\text{copies}})/\Sigma_m\right),$$

with Σ_m acting diagonally. As an example take $m = 2$. In this case the stable factors are given by

$$(\mathbb{S}^1)^{\wedge r}/\Sigma_2 \cong \mathbb{S}^r/\Sigma_2,$$

with the nontrivial element $\tau \in \Sigma_2$ acting on a point $(x_0, ..., x_r) \in \mathbb{S}^r$ by

$$\tau \cdot (x_0, ..., x_r) = (x_0, -x_1, ..., -x_r).$$

The definition of symmetric products is recalled next.

Definition 6.3. Let X be a topological space. The m-th symmetric product of X is defined to be the quotient space

$$SP^m(X) := X^m/\Sigma_m,$$

where the symmetric group Σ_m acts by permuting the different factors of X.

When $m = 0$, the space $SP^0(X)$ is defined to be a point. In general, $SP^m(X)$ can be thought of as the space of unordered m-tuples $[y_1, ..., y_m]$. Suppose that X is a based space with base point x_0. In this situation, there is a natural map

$$SP^m(X) \to SP^{m+1}(X)$$

that sends the unordered m-tuple $[y_1, ..., y_m]$ to $[y_1, ..., y_m, x_0]$. This induces a sequence

$$X = SP^1(X) \to SP^2(X) \to \cdots \to SP^m(X) \to \cdots$$

and $SP^\infty(X)$ is defined as the colimit of this sequence. Notice that $SP^\infty(X)$ is precisely the free abelian monoid generated by X.

Symmetric products naturally appear in the study of spaces of representations; this is explained in the following proposition.

Proposition 6.4. *There are homeomorphisms*

$$Rep(\mathbb{Z}^n, U(m)) \cong SP^m((\mathbb{S}^1)^n),$$
$$Rep(\mathbb{Z}^n, Sp(m)) \cong SP^m((\mathbb{S}^1)^n/\mathbb{Z}/2),$$

where $\mathbb{Z}/2$ acts by complex conjugation on $\mathbb{S}^1$ and diagonally on the torus $(\mathbb{S}^1)^n$. These homeomorphisms are compatible with the standard inclusions

$$U(m) \to U(m+1) \text{ and } Sp(m) \to Sp(m+1).$$

Proof: By Theorem 6.1 when G is a compact Lie group such that $\mathrm{Rep}(\mathbb{Z}^n, G)$ is path-connected for all $n \geq 1$ there is a homeomorphism

$$\mathrm{Rep}(\mathbb{Z}^n, G) \cong T^n/W,$$

where $T \subset G$ is a maximal torus and W is the Weyl group. As pointed out before $\mathrm{Rep}(\mathbb{Z}^n, G)$ is path-connected for every $n \geq 1$ when $G = U(m)$ or $Sp(m)$. For $G = U(m)$ a maximal torus T is of rank m and can be taken to be the space of diagonal matrices with entries in $\mathbb{S}^1$ and the Weyl group $W = \Sigma_m$ acts by permuting the diagonal entries. Therefore

$$\mathrm{Rep}(\mathbb{Z}^n, U(m)) \cong T^n/W \cong ((\mathbb{S}^1)^m)^n/\Sigma_m.$$

Here Σ_m acts diagonally on the previous product and permutes the diagonal factors in each factor $(\mathbb{S}^1)^m$. Note that

$$((\mathbb{S}^1)^m)^n/\Sigma_m \cong ((\mathbb{S}^1)^n)^m/\Sigma_m,$$

where Σ_m acts by permuting the $(\mathbb{S}^1)^n$ factors in the product $((\mathbb{S}^1)^n)^m$. Therefore

$$\mathrm{Rep}(\mathbb{Z}^n, U(m)) \cong ((\mathbb{S}^1)^n)^m/\Sigma_m \cong SP^m((\mathbb{S}^1)^n).$$

On the other hand, for $G = Sp(m)$ a maximal torus T is homeomorphic to $(\mathbb{S}^1)^m$ and the Weyl group W is a semi-direct product

$$W = \Sigma_m \ltimes (\mathbb{Z}/2)^m,$$

where Σ_m acts permuting the factors in $(\mathbb{S}^1)^m$ and if $\mathbb{Z}/2$ acts on $\mathbb{S}^1$ by complex conjugation, then given $(\tau_1, ..., \tau_m) \in (\mathbb{Z}/2)^m$ and $(x_1, ..., x_m) \in (\mathbb{S}^1)^m$ then

$$(\tau_1, ..., \tau_m) \cdot (x_1, ..., x_m) = (\tau_1 \cdot x_1, ..., \tau_m \cdot x_m).$$

Therefore

$$\mathrm{Rep}(\mathbb{Z}^n, Sp(m)) \cong T^n/W \cong ((\mathbb{S}^1)^m)^n/W,$$

with W acting diagonally. In this case

$$((\mathbb{S}^1)^m)^n/W \cong ((\mathbb{S}^1)^n)^m/W \cong ((\mathbb{S}^1)^n/\mathbb{Z}/2)^m/\Sigma_m \cong SP^m((\mathbb{S}^1)^n/\mathbb{Z}/2).$$

□

Remark. The quotient space $(\mathbb{S}^1)^n/\mathbb{Z}/2$ is a singular space with 2^n isolated singularities. Neighborhoods around the singular points look like cones on $\mathbb{R}P^{n-1}$. For example, when $n = 4$ the quotient $(\mathbb{S}^1)^4/\mathbb{Z}/2$ is an orbifold with 16 isolated singularities.

The identification of $\mathrm{Rep}(\mathbb{Z}^n, G)$ as a symmetric product for $G = U(m)$ and $Sp(m)$ has interesting consequences. For example the following can be derived.

Corollary 6.5. *For every $m \geq 1$ there is a homeomorphism*

$$Rep(\mathbb{Z}^2, Sp(m)) \cong \mathbb{C}P^m.$$

Proof: By the previous corollary

$$\mathrm{Rep}(\mathbb{Z}^2, Sp(m)) \cong SP^m((\mathbb{S}^1 \times \mathbb{S}^1)/\mathbb{Z}/2).$$

The quotient space $(\mathbb{S}^1 \times \mathbb{S}^1)/\mathbb{Z}/2$ is homeomorphic to $\mathbb{S}^2$. Therefore

$$\mathrm{Rep}(\mathbb{Z}^2, Sp(m)) \cong SP^m(\mathbb{S}^2) \cong \mathbb{C}P^m.$$

□

Suppose now that G is a topological abelian group. The multiplication on G defines a continuous function

$$\begin{aligned} \mu_m : SP^m(G) &\to G \\ [x_1, ..., x_m] &\mapsto x_1 \cdots x_m. \end{aligned}$$

When $G = (\mathbb{S}^1)^n$ the following diagram commutes

$$\begin{tikzcd} SP^m((\mathbb{S}^1)^n) \arrow[r, "\mu_m"] \arrow[d, "\cong"'] & (\mathbb{S}^1)^n \\ \mathrm{Rep}(\mathbb{Z}^n, U(m)) \arrow[ur, "det_*"'] & \end{tikzcd}$$

and thus μ_m can be identified with the map induced by the determinant det_*. Note that $det_*^{-1}(1, ..., 1)$ is precisely the space of commuting n-tuples of elements in $SU(m)$ modulo conjugation in $U(m)$. This space agrees with $\mathrm{Rep}(\mathbb{Z}^n, SU(m))$ because any two elements in $\mathrm{Hom}(\mathbb{Z}^n, SU(m))$ are conjugate by an element in $U(m)$ if and only if they are conjugate by an element in $SU(m)$. Moreover the map

$$\mu_m : SP^m((\mathbb{S}^1)^n) \to (\mathbb{S}^1)^n$$

is a locally trivial fiber bundle. This can be seen by picking an n-th root of the determinant function which can always be done locally. For the particular case of $n = 2$, the map μ_m agrees with the Abel-Jacobi map

$$\mu_m : SP^m(\mathbb{S}^1 \times \mathbb{S}^1) \to \mathbb{S}^1 \times \mathbb{S}^1.$$

In general, if $\mathcal{M}_g$ is a Riemann surface of genus g and $J(\mathcal{M}_g)$ denotes its Jacobian then the Abel-Jacobi is a map

$$\mu_m : SP^m(\mathcal{M}_g) \to J(\mathcal{M}_g),$$

that makes $SP^m(\mathcal{M}_g)$ into a fiber bundle over $J(\mathcal{M}_g)$ with fiber type $\mathbb{C}P^{m-g}$ for $m \geq 2g$. In particular $\mathrm{Rep}(\mathbb{Z}^2, SU(m)) \cong \mathbb{C}P^{m-1}$ for $m \geq 2$. This proves the following proposition.

Proposition 6.6. *For every n and $m \geq 1$ composition with the determinant defines a locally trivial fiber bundle*

$$det_* : Rep(\mathbb{Z}^n, U(m)) \to (\mathbb{S}^1)^n.$$

with fiber type $Rep(\mathbb{Z}^n, SU(m))$. In particular, for $m \geq 2$

$$Rep(\mathbb{Z}^2, SU(m)) \cong \mathbb{C}P^{m-1}.$$

The identification $\mathrm{Rep}(\mathbb{Z}^2, U(m)) \cong SP^m(\mathbb{S}^1 \times \mathbb{S}^1)$ can also be used to prove the following corollary using the results from [18].

Corollary 6.7.

$$H^i(Rep(\mathbb{Z}^2, U(m)); \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \textit{if} \quad i = 0, \\ \mathbb{Z} \oplus \mathbb{Z} & \textit{if} \quad 1 \leq i \leq 2m-1, \\ \mathbb{Z} & \textit{if} \quad i = 2m, \\ 0 & \textit{if} \quad i > 2m. \end{cases}$$

In general the cohomology groups of the spaces $\mathrm{Rep}(\mathbb{Z}^n, U(m))$ and $\mathrm{Rep}(\mathbb{Z}^n, Sp(m))$ can be computed using the identifications given in Proposition 6.4. Note that by [25, Section 22] the inclusions

$$\mathrm{Rep}(\mathbb{Z}^n, U(m)) \to \mathrm{Rep}(\mathbb{Z}^n, U(m+1)) \text{ and } \mathrm{Rep}(\mathbb{Z}^n, Sp(m)) \to \mathrm{Rep}(\mathbb{Z}^n, Sp(m+1))$$

split on the level of cohomology.

Definition 6.8. For each $n \geq 1$, define $\mathrm{Rep}(\mathbb{Z}^n, SU)$ to be the colimit of the sequence

$$\mathrm{Rep}(\mathbb{Z}^n, SU(1)) \to \mathrm{Rep}(\mathbb{Z}^n, SU(2)) \to \cdots \to \mathrm{Rep}(\mathbb{Z}^n, SU(m)) \to \mathrm{Rep}(\mathbb{Z}^n, SU(m+1)) \cdots$$

induced by the canonical inclusions $SU(m) \to SU(m+1)$. Similarly, $\mathrm{Rep}(\mathbb{Z}^n, U)$ and $\mathrm{Rep}(\mathbb{Z}^n, Sp)$ are defined as the colimit of the finite stages under the canonical inclusions.

The homotopy type of these spaces is established in the following theorem[1]

Theorem 6.9. *For every $n \geq 1$ there are homotopy equivalences*

$$\begin{aligned} Rep(\mathbb{Z}^n, SU) &\simeq \prod_{2 \leq i \leq n} K(\mathbb{Z}^{\binom{n}{i}}, i), \\ Rep(\mathbb{Z}^n, U) &\simeq \prod_{1 \leq i \leq n} K(\mathbb{Z}^{\binom{n}{i}}, i), \\ Rep(\mathbb{Z}^n, Sp) &\simeq \prod_{1 \leq i \leq \lfloor n/2 \rfloor} K(\mathbb{Z}^{\binom{n}{2i}} \oplus (\mathbb{Z}/2)^{r(2i)}, 2i), \end{aligned}$$

where the $r(i)$'s are integer defined by

$$r(i) = \begin{cases} \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n-i-1} & \text{if} \quad 1 \leq i \leq n \\ 0 & \text{if} \quad i > n. \end{cases}$$

Proof: By Proposition 6.4 there are homeomorphisms

$$\begin{aligned} \mathrm{Rep}(\mathbb{Z}^n, U(m)) &\cong SP^m((\mathbb{S}^1)^n), \\ \mathrm{Rep}(\mathbb{Z}^n, Sp(m)) &\cong SP^m((\mathbb{S}^1)^n/\mathbb{Z}/2), \end{aligned}$$

which are compatible with the standard inclusions. It follows that there are homeomorphisms

$$\mathrm{Rep}(\mathbb{Z}^n, U) \cong SP^\infty((\mathbb{S}^1)^n) \text{ and } \mathrm{Rep}(\mathbb{Z}^n, Sp) \cong SP^\infty((\mathbb{S}^1)^n/\mathbb{Z}/2).$$

In general in [14] it is proved that if X is a connected CW complex of finite type then

$$SP^\infty(X) \simeq \prod_{n \geq 1} K(H_n(X, \mathbb{Z}), n).$$

This is then used to handled the cases of U and Sp. (The homology groups of $(\mathbb{S}^1)^n/\mathbb{Z}/2$ are computed below in proposition 6.10). To handle the case of SU notice that a homotopy equivalence

$$f : \prod_{1 \leq i \leq n} K(\mathbb{Z}^{\binom{n}{i}}, i) \to SP^\infty((\mathbb{S}^1)^n) \cong \mathrm{Rep}(\mathbb{Z}^n, U)$$

[1] A proof for the case of the unitary groups also appears in [22, Theorem 6.6].

can be constructed as follows. For each $1 \le i \le n$ pick

$$f_i : \bigvee^{\binom{n}{i}} \mathbb{S}^i \to SP^\infty((\mathbb{S}^1)^n)$$

inducing an isomorphism in π_i for $2 \le i \le n$ and an isomorphism on H_1 for $i = 1$. The map

$$f_1 : \bigvee^{n} \mathbb{S}^1 \to SP^\infty((\mathbb{S}^1)^n)$$

can be chosen to be the inclusion

$$\bigvee^{n} \mathbb{S}^1 \to (\mathbb{S}^1)^n = SP^1((\mathbb{S}^1)^n) \to SP^\infty((\mathbb{S}^1)^n)$$

where the j-th wedge factor in $\bigvee^n \mathbb{S}^1$ is mapped into the j-th factor of $(\mathbb{S}^1)^n$. Since $SP^\infty((\mathbb{S}^1)^n)$ has the structure of a monoid the different $f_i's$ can be assembled to get a homomorphism

$$g : SP^\infty(\bigvee_{1\le i\le n} \bigvee^{\binom{n}{i}} \mathbb{S}^i) \to \mathrm{Rep}(\mathbb{Z}^n, U).$$

This map is a homotopy equivalence. Also there is a homotopy equivalence

$$h : (\mathbb{S}^1)^n \times \prod_{2\le i\le n} K(\mathbb{Z}^{\binom{n}{i}}, i) = \prod_{1\le i\le n} SP^\infty(\bigvee^{\binom{n}{i}} \mathbb{S}^i) \cong SP^\infty(\bigvee_{1\le i\le n} \bigvee^{\binom{n}{i}} \mathbb{S}^i),$$

and f is defined to be the composition $f = g \circ h$. On the other hand, the locally trivial fibration sequences

$$\mathrm{Rep}(\mathbb{Z}^n, SU(m)) \to \mathrm{Rep}(\mathbb{Z}^n, U(m)) \overset{det_*}{\to} (\mathbb{S}^1)^n$$

are compatible with the standard inclusions $SU(m) \to SU(m+1)$ and $U(m) \to U(m+1)$. Therefore the determinant gives rise to a well defined function

$$det_* : \mathrm{Rep}(\mathbb{Z}^n, U) \to (\mathbb{S}^1)^n$$

which is a fibration with fiber type $\mathrm{Rep}(\mathbb{Z}^n, SU)$. Moreover, by the way the map f_1 was chosen, it follows that there is a commutative diagram of fibrations

$$\begin{array}{ccc}
\prod_{2\le i\le n} K(\mathbb{Z}^{\binom{n}{i}}, i) & \xrightarrow{f_|} & \mathrm{Rep}(\mathbb{Z}^n, SU) \\
\downarrow & & \downarrow \\
(\mathbb{S}^1)^n \times \prod_{2\le i\le n} K(\mathbb{Z}^{\binom{n}{i}}, i) & \xrightarrow{f} & \mathrm{Rep}(\mathbb{Z}^n, U) \\
{\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle det_*} \\
(\mathbb{S}^1)^n & \xrightarrow{=} & (\mathbb{S}^1)^n
\end{array}$$

From the long exact sequences on homotopy groups associated to these fibrations and the five lemma it follows that

$$f_| : \prod_{2\le i\le n} K(\mathbb{Z}^{\binom{n}{i}}, i) \to \mathrm{Rep}(\mathbb{Z}^n, SU)$$

is a weak homotopy equivalence. The proposition follows by noting that $\mathrm{Rep}(\mathbb{Z}^n, SU)$ has the homotopy type of a CW complex. $\square$

Proposition 6.10. *The homology groups of $(\mathbb{S}^1)^n/\mathbb{Z}/2$ are given by*

$$H_i((\mathbb{S}^1)^n/\mathbb{Z}/2;\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{if} \quad i = 0, \\ \mathbb{Z}^{\binom{n}{i}} \oplus (\mathbb{Z}/2)^{\binom{n}{0}+\binom{n}{1}+\cdots+\binom{n}{n-i-1}} & \text{if} \quad i \text{ is even and } 0 < i \leq n, \\ 0 & \text{else.} \end{cases}$$

Proof: Let $G = \mathbb{Z}/2$ and $X = (\mathbb{S}^1)^n$ seen as a G-space with G acting diagonally and by complex conjugation on each $\mathbb{S}^1$ factor. Denote by F the fixed point set under this G action which in this case is a discrete set with 2^n points. The cohomology groups $H^i(X/G;\mathbb{Z})$ are computed first. Using the long exact sequence in cohomology associated to the pair $(X/G, F)$, it suffices to compute the relative cohomology groups $H^*(X/G, F;\mathbb{Z})$. To do so, consider the map $\varphi : X \times_G EG \to X/G$ induced by the G-equivariant projection $\pi_1 : X \times EG \to X$. By [12, Proposition 1.1 VII] the map φ induces an isomorphism

$$\varphi^* : H^*(X/G, F;\mathbb{Z}) \to H^*_G(X, F;\mathbb{Z}). \tag{4}$$

The equivariant cohomology groups $H^*_G(X, F;\mathbb{Z})$ are computed next. This is achieved by studying the map $j^*_G : H^*_G(X;\mathbb{Z}) \to H^*_G(F;\mathbb{Z})$ induced by the inclusion $j : F \hookrightarrow X$. Consider the fibration sequence

$$X \to X \times_G EG \to BG.$$

By [7, Theorem 1.2] the Lyndon-Hochshild-Serre spectral sequence associated to this fibration collapses on the E_2-term and there are no extension problems. Therefore

$$H^k_G(X;\mathbb{Z}) = H^k(X \times_G EG;\mathbb{Z}) \cong \bigoplus_{i+j=k} H^i(G; H^j(X;\mathbb{Z})).$$

Using [1, Proposition 1.10] these cohomology groups can be computed explicitly to obtain

$$H^k_G(X;\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{if} \quad k = 0, \\ \mathbb{Z}^{\binom{n}{k}} \oplus (\mathbb{Z}/2)^{\binom{n}{0}+\cdots+\binom{n}{k-1}} & \text{if} \quad k \text{ is even and } 0 < k \leq n, \\ (\mathbb{Z}/2)^{2^n} & \text{if} \quad k \text{ is even }, k > n, \\ 0 & \text{if} \quad k \text{ is odd.} \end{cases} \tag{5}$$

In particular, the map $j^*_G : H^k_G(X;\mathbb{Z}) \to H^k_G(F;\mathbb{Z})$ is trivial for k odd. On the other hand, since G acts trivially on F, then $F \times_G EG = F \times BG$ and thus

$$H^k_G(F;\mathbb{Z}) = H^k(F \times BG) \cong \begin{cases} \mathbb{Z}^{2^n} & \text{if} \quad k = 0, \\ (\mathbb{Z}/2)^{2^n} & \text{if} \quad k \text{ is even}, k > 0, \\ 0 & \text{if} \quad k \text{ is odd.} \end{cases} \tag{6}$$

By the universal coefficient theorem the natural map

$$H^k(X \times_G EG;\mathbb{Z}) \otimes \mathbb{Z}/2 \to H^k(X \times_G EG;\mathbb{Z}/2)$$

is an isomorphism when k is even since in this case the Tor term vanishes as

$$H^{k+1}(X \times_G EG;\mathbb{Z}) = 0 \text{ when } k \text{ is even.}$$

The same is true when X is replaced by F. Therefore when k is even there is a commutative diagram

$$\begin{tikzcd} H^k(X\times_G EG;\mathbb{Z})\otimes\mathbb{Z}/2 \arrow[r, "\cong"] \arrow[d, "j_G^*\otimes\mathbb{Z}/2"'] & H^k(X\times_G EG;\mathbb{Z}/2) \arrow[d, "j^*_{G,\mathbb{Z}/2}"] \\ H^k(F\times_G EG;\mathbb{Z})\otimes\mathbb{Z}/2 \arrow[r, "\cong"] & H^k(F\times_G EG;\mathbb{Z}/2). \end{tikzcd} \tag{7}$$

On the other hand, since $H^k(X;\mathbb{Z}/2)=0$ for $k>n$, then by [12, Theorem 1.5 VII] the map

$$j^*_{G,\mathbb{Z}/2}: H^k_G(X;\mathbb{Z}/2)\to H^k_G(F;\mathbb{Z}/2)$$

is an isomorphism for $k>n$. Moreover, for any $k\geq 0$ the map $j^*_{G,\mathbb{Z}/2}$ is injective. To see this note that

$$\sum_{i\geq 0} rkH^k(X,\mathbb{Z}/2)=\sum_{i\geq 0} rkH^k(F,\mathbb{Z}/2)=2^n.$$

Therefore, by [12, Theorem 1.6 VII] X is totally nonhomologous to zero (mod p) [1] in $X\times_G EG$ and by [12, Theorem 1.5 VII] the map $j^*_{G,\mathbb{Z}/2}$ is injective. Notice that when k is even by (6) it follows that $H^k_G(F;\mathbb{Z})\otimes\mathbb{Z}/2\cong H^k_G(F;\mathbb{Z})$. Using this and (7) it follows that when k is even the image of

$$j^*_G: H^k_G(X;\mathbb{Z})\to H^k_G(F;\mathbb{Z})$$

is a $\mathbb{Z}/2$-vector space of rank $\binom{n}{0}+\binom{n}{1}+\cdots+\binom{n}{k}$. Similarly, using (5), (6) and (7) it can be seen that when k is even $\mathrm{Ker}(j^*_G)\cong\mathbb{Z}^{\binom{n}{k}}$. By the long exact sequence in G-equivariant cohomology associated to the pair (X,F) if follows that

$$H^k_G(X,F;\mathbb{Z})\cong\begin{cases} 0 & \text{if} \quad k=0,\\ \mathbb{Z}^{2^n-1} & \text{if} \quad k=1,\\ \mathbb{Z}^{\binom{n}{k}} & \text{if} \quad k \text{ is even}, 0<k\leq n,\\ (\mathbb{Z}/2)^{\binom{n}{0}+\binom{n}{1}+\cdots+\binom{n}{n-k}} & \text{if} \quad k \text{ is odd}, 1<k\leq n,\\ 0 & \text{if} \quad k>n. \end{cases}$$

Since F is discrete, from the long exact sequence in cohomology associated to the pair $(X/G,F)$ it follows that $H^k(X/G,F;\mathbb{Z})\cong H^k(X/G;\mathbb{Z})$ for $k\geq 2$. It is easy to see that $H^0(X/G;\mathbb{Z})=\mathbb{Z}$ and $H^1(X/G;\mathbb{Z})=0$. Using this and the isomorphism (4) the cohomology groups $H^k(X/G;\mathbb{Z})$ are computed. Finally, the result follows by applying the universal coefficient theorem to obtain the homology groups $H_i(X/G;\mathbb{Z})$. □

On the other hand, as noted before a stable splitting for the spaces of the from $\mathrm{Rep}(\mathbb{Z}^n,G)$ is obtained by Theorem 5.6. Via the identifications

$$\mathrm{Rep}(\mathbb{Z}^n,U(m))\cong SP^m((\mathbb{S}^1)^n) \text{ and } \mathrm{Rep}(\mathbb{Z}^n,Sp(m))\cong SP^m((\mathbb{S}^1)^n/\mathbb{Z}/2)$$

a stable splitting for the spaces $SP^m((\mathbb{S}^1)^n)$ and $SP^m((\mathbb{S}^1)^n/\mathbb{Z}/2)$ is then obtained. It turns out that a similar splitting holds in general for spaces of the form $SP^m(G^n)$ when G is a topological group with nondegenerate unit. This can be seen using the following definition given by May [19, Definition A.7].

[1] A G-space X is said to be totally nonhomologous to zero (mod p) if the restriction to a typical fiber $H^*(X\times_G EG;\mathbb{Z}/p)\to H^*(X;\mathbb{Z}/p)$ is surjective.

Definition 6.11. Let $\mathcal{U}$ be the category of compactly generated Hausdorff spaces. A functor $F : \mathcal{U} \to \mathcal{U}$, is said to be admissible if for any NDR representation of (h, u) of (X, A) as an NDR pair determines a representation (Fh, Fu) of (FX, FA) as an NDR pair such that $(Fh)_t = F(h_t)$ and such that, for any map $g : X \to X$ that satisfies the condition $u(g(x)) < 1$ whenever $u(x) < 1$, then the map $Fu : FX \to [0, 1]$ satisfies $(Fu)(Fg)(y) < 1$ whenever $Fu(y) < 1$, $y \in FX$.

For every $m \geq 1$ the functor SP^m is an admissible functor. Indeed, if (X, A) is an NDR pair represented by (h, u), then $(SP^m X, SP^m A)$ is an NDR pair represented by $(SP^m h, SP^m u)$, where

$$SP^m h(([x_1, .., x_m], t)) = [h(x_1, t), ..., h(x_m, t)]$$
$$SP^m u([x_1, .., x_m]) = \max_{1 \leq i \leq m} u(x_i).$$

The other condition of an admissible functor is easily verified. From the definition it follows that if F is an admissible functor and X_* is a simplicial space that is simplicially NDR (resp. proper), then the simplicial space FX_* is also simplicially NDR (resp. proper). In particular, a stable decomposition as in Theorem 5.2 for FX_n is obtained whenever this theorem applies for X_n. Therefore if G is a topological group with nondegenerate unit; that is; $(G, \{1_G\})$ is an NDR pair, then the simplicial space B_*G is simplicially NDR and it is proper if $(G, \{1_G\})$ is further assumed to be a strong NDR pair. Let $S^1(SP^m(G^n))$ be the subspace of $SP^m(G^n)$ consisting of unordered m-tuples $\underline{x} = [x_1, ..., x_m]$ with $x_i \in G^n$ with the following property. If

$$x_i = (g_{i1}, ..., g_{in}),$$

then there is a j such that $g_{ij} = 1_G$ for all $1 \leq i \leq n$. If the elements g_{ij} are seen as the entries of an $m \times n$ matrix $A(\underline{x})$, then this condition means that at least one of the columns of $A(\underline{x})$ has entries all equal to 1_G. From the previous remark the following corollary is obtained.

Corollary 6.12. *For any topological group G with nondegenerate unit and any n and $m \geq 1$ there is a natural homotopy equivalence*

$$\Sigma SP^m(G^n) \simeq \bigvee_{1 \leq r \leq n} \Sigma \left(\bigvee^{\binom{n}{r}} SP^m(G^r)/S^1(SP^m(G^r)) \right).$$

If in addition $(G, \{1\})$ is a strong NDR pair, then each factor $SP^m(G^r)/SP^m(F^1G^r)$ is stably equivalent to the quotient $F_rSP^mBG/F_{r-1}SP^mBG$, for a filtration of SP^mBG

$$F_0SP^mBG \subset F_1SP^mBG \subset \cdots \subset F_rSP^mBG \subset \cdots \subset SP^mBG.$$

Proof: From the previous remark, the simplicial space $SP^m(G^*)$ is simplicially NDR and proper if $(G, \{1\})$ is a strong NDR pair. By Theorem 5.2, the filtration

$$\emptyset = S^{n+1}(SP^m(G^n)) \subset S^n(SP^m(G^n)) \subset \cdots \subset S^1(SP^m(G^n)) \subset SP^m(G^n)$$

of $SP^m(G^n)$ given by the degeneracy maps splits after one suspension and there is a natural homotopy equivalence

$$\Sigma SP^m(G^n) \simeq \bigvee_{0 \leq r \leq n} \Sigma(S^r(SP^m(G^n))/S^{r+1}(SP^m(G^n))).$$

The first part of the corollary follows by showing that

$$S^r(SP^m(G^n))/S^{r+1}(SP^m(G^n)) \cong \bigvee^{\binom{n}{n-r}} SP^m(G^{n-r})/S^1(SP^m(G^{n-r})).$$

This is achieved in the same way as in the case of the spaces $B_n(G,K)$ using the different projection maps

$$\begin{aligned} P_{m_1,\ldots,m_{n-r}} : G^n &\to G^{n-r} \\ (x_1,\ldots,x_n) &\mapsto (x_{m_1},\ldots,x_{m_{n-r}}). \end{aligned}$$

for all possible sequences $1 \leq m_1 \leq \cdots m_{n-r} \leq n$. To show the last part note that if $(G,\{1_G\})$ is a strong NDR pair, then B_*G is a proper simplicial space. By [19, Lemma 11.3] each factor $SP^m(G^r)/S^1(SP^m(G^r))$ is stably equivalent to the quotient $F_r|SP^mB_*G|/F_{r-1}|SP^mB_*G|$, where

$$F_0|SP^mB_*G| \subset F_1|SP^mB_*G| \subset \cdots \subset F_r|SP^mB_*G| \subset \cdots |SP^mB_*G|$$

is the natural filtration of $|SP^mB_*G|$. But in the category of compactly generated weak Hausdorff spaces $|SP^mB_*G| \cong SP^mBG$. □

7. Compact Lie groups of rank one

In this section the stable factors appearing in Theorem 5.6 are studied for the particular case of compact connected rank one Lie groups.

Let G be a compact, connected rank one Lie group. If G is such a Lie group then it is isomorphic to $\mathbb{S}^1$, $SU(2)$ or $SO(3)$. A similar argument can be used to understand the stable factors $B_n(G,K)/S_n^1(G,K)$ in these three cases. The case $K=\{1_G\}$ is handled first, this corresponds to the space of commuting n-tuples in G. Fix T a maximal torus for G and let W denote the corresponding Weyl group. When $G=\mathbb{S}^1$ then $T=\mathbb{S}^1$, for $G=SU(2)$ the maximal torus T can be taken to be the space of diagonal matrices with entries in $\mathbb{S}^1$ and determinant one and when $G=SO(3)$, T can be taken to be the space of rotations with respect to the z-axis. The possibilities for W are the trivial group when $G=\mathbb{S}^1$ or Σ_2 acting on the Lie algebra $\mathfrak{t} \cong i\mathbb{R}$ by complex conjugation in the other cases. In any of these cases there is a W-equivariant homeomorphism

$$\psi : \mathfrak{t} \to T - \{1\}$$

sending 0 to $-1 \in T$. When $G=\mathbb{S}^1$, ψ can be taken to be the transformation

$$\begin{aligned} \psi : \mathfrak{t} \cong i\mathbb{R} &\to T - \{1\} \\ z &\mapsto \frac{z+1}{z-1}. \end{aligned}$$

When $G=SU(2)$, the map ψ can be taken to be

$$\begin{aligned} \psi : \mathfrak{t} \cong i\mathbb{R} &\to T - \{1\} \\ z &\mapsto \begin{pmatrix} \frac{z+1}{z-1} & \\ & \frac{z-1}{z+1} \end{pmatrix}, \end{aligned}$$

and similarly for $SO(3)$.

The main tool for understanding the stable factors $\mathrm{Hom}(\mathbb{Z}^n, G)/S^1_n(G)$ is the map already used in Theorem 6.1

$$\begin{aligned} \bar{\varphi}_n : G/T \times_W T^n &\to \mathrm{Hom}(\mathbb{Z}^n, G)_{(1,...,1)} \\ (\bar{g}, t_1, ..., t_n) &\mapsto (gt_1g^{-1}, ..., gt_ng^{-1}), \end{aligned}$$

where $\mathrm{Hom}(\mathbb{Z}^n, G)_{(1,...,1)}$ denotes the path-connected component of $\mathrm{Hom}(\mathbb{Z}^n, G)$ containing the n-tuple $(1, ..., 1)$. Note that trivially $\mathrm{Hom}(\mathbb{Z}^1, G)/S^1_1(G) = G$ and thus from now on n is assume to be at least 2. As pointed out before, $\bar{\varphi}_n$ is a continuous surjective map. Also, if $g \in G$ and $t \in T$, then $gtg^{-1} = 1$ if and only if $t = 1$. This shows that the restriction of $\bar{\varphi}_n$ defines a surjective continuous map

$$\alpha_n = (\bar{\varphi}_n)_| : G/T \times_W (T - \{1\})^n \to (\mathrm{Hom}(\mathbb{Z}^n, G)_{(1,...,1)} - S^1_n(G)_{(1,...,1)}),$$

where $S^1_n(G)_{(1,..,1)} = S^1_n(G) \cap \mathrm{Hom}(\mathbb{Z}^n, G)_{(1,...,1)}$. Since the inclusion map

$$S^1_n(G) \hookrightarrow \mathrm{Hom}(\mathbb{Z}^n, G)$$

is a cofibration, in particular

$$S^1_n(G)_{(1,...,1)} \hookrightarrow \mathrm{Hom}(\mathbb{Z}^n, G)_{(1,...,1)}$$

is also a cofibration, it follows that

$$\mathrm{Hom}(\mathbb{Z}^n, G)_{(1,...,1)}/S^1_n(G)_{(1,...,1)} \cong (\mathrm{Hom}(\mathbb{Z}^n, G)_{(1,...,1)} - S^1_n(G)_{(1,...,1)})^+,$$

where in general for a space X, X^+ denotes its one point compactification. On the other hand, using the W-equivariant homeomorphism ψ, it follows that

$$G/T \times_W (T - \{1\})^n \cong G/T \times_W \mathfrak{t}^n.$$

Note that W acts freely on G/T (this is the induced action of G acting on itself on the right) and thus the projection map $p_n : G/T \times_W \mathfrak{t}^n \to (G/T)/W$ is a vector bundle over $(G/T)/W = G/N(T)$. The map $\bar{\varphi}_n$ and its restriction α_n are not injective for a general Lie group. The different possibilities are considered next.

When $G = \mathbb{S}^1$, the map α_n is a homeomorphism and $G/N(T) = *$. Since $\mathrm{Hom}(\mathbb{Z}^n, \mathbb{S}^1)$ is connected, then $\mathrm{Hom}(\mathbb{Z}^n, \mathbb{S}^1)/S^1_n(\mathbb{S}^1)$ is homeomorphic to the one point compactification of $\mathfrak{t}^n$; that is, in this case the stable factors are given by

$$\mathrm{Hom}(\mathbb{Z}^n, \mathbb{S}^1)/S^1_n(\mathbb{S}^1) \cong \mathbb{S}^n.$$

Of course, this can be seen trivially using the fact that $\mathrm{Hom}(\mathbb{Z}^n, \mathbb{S}^1) = (\mathbb{S}^1)^n$ as $\mathbb{S}^1$ is abelian. Modulo conjugation the situation is trivial since $\mathbb{S}^1$ is abelian, thus

$$\mathrm{Rep}(\mathbb{Z}^n, \mathbb{S}^1)/\bar{S}^1_n(\mathbb{S}^1) \cong \mathbb{S}^n.$$

When G equals $SU(2)$ the map α_n is no longer injective. Let

$$s_n : G/N(T) \to G/T \times_W \mathfrak{t}^n$$

be the zero section of the vector bundle p_n. Then in this case

$$\alpha_n^{-1}(-1, ..., -1) = s_n(G/N(T))$$

and $\alpha_n^{-1}(g_1, ..., g_n)$ is a point for $(g_1, ..., g_n) \neq (-1, ..., -1)$. On the other hand, the space $\mathrm{Hom}(\mathbb{Z}^n, SU(2))$ is connected, $G/T \cong \mathbb{S}^2$ with W acting as the antipodal map and therefore

$G/N(T) \cong \mathbb{R}P^2$. If λ_n denotes the canonical vector bundle over $\mathbb{R}P^n$, then the vector bundle $p_n : G/T \times_W \mathfrak{t}^n \to (G/T)/W$ is precisely $n\lambda_2$, the Whitney sum of n copies of λ_2. Therefore

$$\mathrm{Hom}(\mathbb{Z}^n, SU(2))/S_n^1(SU(2)) \cong \begin{cases} \mathbb{S}^3 & \text{if } n = 1, \\ (\mathbb{R}P^2)^{n\lambda_2}/s_n(\mathbb{R}P^2) & \text{if } n \geq 2 \end{cases}$$

where X^μ denotes the associated Thom space for a vector bundle μ over X. This agrees with the computations given independently in [9] and [13]. Modulo conjugation the situation simplifies. As pointed out before

$$\mathrm{Rep}(\mathbb{Z}^n, SU(2))/\bar{S}_n^1(SU(2)) \cong \mathbb{S}^n/\Sigma_2$$

with the nontrivial element $\tau \in \Sigma_2$ acting on a point $(x_0, ..., x_n) \in \mathbb{S}^r$ by

$$\tau \cdot (x_0, ..., x_r) = (x_0, -x_1, ..., -x_r).$$

Suppose now that $G = SO(3)$, then $\mathrm{Hom}(\mathbb{Z}^n, SO(3))$ is no longer connected. In this case $G/T \cong \mathbb{S}^2$ with W acting as the antipodal map and therefore as in the case of $SU(2)$, $G/N(T) \cong \mathbb{R}P^2$ and $p_n : G/T \times_W \mathfrak{t}^n \to (G/T)/W$ is the vector bundle $n\lambda_2$. In this case, the map α_n is a homeomorphism and therefore

$$\mathrm{Hom}(\mathbb{Z}^n, SO(3))_{(1,...,1)}/S_n^1(SO(3))_{(1,...,1)} \cong (\mathbb{R}P^2)^{n\lambda_2}.$$

On the other hand, by [23]

$$\mathrm{Hom}(\mathbb{Z}^n, SO(3)) \cong \mathrm{Hom}(\mathbb{Z}^n, SO(3))_{(1,...,1)} \sqcup \left(\bigsqcup_{A(n)} \mathbb{S}^3/Q_8 \right), \tag{8}$$

where

$$A(n) := \frac{(2^n - 1)(2^{n-1} - 1)}{3}$$

and Q_8 denotes the quaternion group. Moreover, if $n \geq 2$ and

$$\underline{x} := (x_1, ..., x_n) \in \mathrm{Hom}(\mathbb{Z}^n, SO(3))$$

belongs to a path-connected component different from $\mathrm{Hom}(\mathbb{Z}^n, SO(3))_{(1,...,1)}$, with $x_k = 1$ for some k, then $y_k = 1$ for every commuting sequence $\underline{y} := (y_1, ..., y_n)$ that belongs to the same path-connected component of $\underline{x}$. This shows that under the identification (8) the subspace $S_n^1(SO(3))$ corresponds to

$$S_n^1(SO(3)) \cong S_n^1(SO(3))_{(1,...,1)} \sqcup \left(\bigsqcup_{B(n)} \mathbb{S}^3/Q_8 \right),$$

where $B(n)$ is the number of connected components homeomorphic to $\mathbb{S}^3/Q_8$ containing an n-tuple $(x_1, ..., x_n)$ with at least one $x_i = 1$. It follows then that

$$\mathrm{Hom}(\mathbb{Z}^n, SO(3))/S_n^1(SO(3)) \cong \mathrm{Hom}(\mathbb{Z}^n, SO(3))_{(1,...,1)}/S_n^1(SO(3))_{(1,...,1)} \vee \left(\bigvee_{C(n)} (\mathbb{S}^3/Q_8)_+ \right).$$

The numbers $C(n)$ are such that $C(1) = 0$, $C(2) = 1$ and satisfy the recurrence relation

$$\sum_{r=1}^{n} C(r)\binom{n}{r} = \frac{(2^n-1)(2^{n-1}-1)}{3}$$

whose solution is

$$C(n) = \frac{1}{2}(3^{n-1}-1).$$

Since $SO(3) \cong \mathbb{R}P^3$, it follows that there are homeomorphisms

$$\mathrm{Hom}(\mathbb{Z}^n, SO(3))/S_n^1(SO(3)) \cong \begin{cases} \mathbb{R}P^3 & \text{if } n = 1, \\ (\mathbb{R}P^2)^{n\lambda_2} \vee \left(\bigvee_{C(n)}(\mathbb{S}^3/Q_8)_+\right) & \text{if } n \geq 2. \end{cases}$$

Modulo conjugation, the group $SO(3)$ acts transitively on each connected component of $\mathrm{Hom}(\mathbb{Z}^n, SO(3))$ that is homeomorphic to $\mathbb{S}^3/Q_8$. Therefore

$$\mathrm{Rep}(\mathbb{Z}^n, SO(3))/\bar{S}_n^1(SO(3)) \cong (\bigvee_{C(n)} \mathbb{S}^0) \vee \mathbb{S}^n/\Sigma_2$$

with Σ_2 acting on $\mathbb{S}^n$ as before. This completely describes the factors $\mathrm{Hom}(\mathbb{Z}^n, G)/S_n^1(G)$ and $\mathrm{Rep}(\mathbb{Z}^n, G)/\bar{S}_n^1(G)$ when G is a rank 1 compact, connected Lie group.

Next, the case of almost commuting elements for G of rank one is discussed. If $G = \mathbb{S}^1$ or $SO(3)$ the spaces of almost commuting elements and commuting elements agree. Thus the only case left to consider is $G = SU(2)$ and $K = Z(SU(2)) \cong \mathbb{Z}/2$. The stable factors $B_n(SU(2), \mathbb{Z}/2)/S_n^1(SU(2), \mathbb{Z}/2)$ can be described in a similar way as in the case of commuting elements. To see this, recall that

$$B_n(SU(2), \mathbb{Z}/2) = \bigsqcup_{C \in T(n, \mathbb{Z}/2)} \mathcal{AC}_{SU(2)}(C).$$

If $C = C_1$ is the trivial matrix whose entries are all equal to 1, then

$$\mathcal{AC}_{SU(2)}(C) = \mathrm{Hom}(\mathbb{Z}^n, SU(2))$$

which is a path-connected space. On the other hand, suppose that $C = (c_{ij})$ is a non trivial matrix with $\mathcal{AC}_{SU(2)}(C)$ not empty. In this case there exist $1 \leq i < j \leq n$ such that $c := c_{ij} \neq 1$. For each $1 \leq k \leq n$ write $c_{ik} = c^{b_k}$ and $c_{jk} = c^{a_k}$ with $0 \leq a_k, b_k \leq 1$. In [5] it is proved that the map

$$\psi : \mathcal{AC}_{SU(2)}(C(c)) \times (\mathbb{Z}/2)^{n-2} \to \mathcal{AC}_{SU(2)}(C) \tag{9}$$

$$((x_i, x_j), (w_1, ..., \hat{w}_i, ..., \hat{w}_j, ..., w_n)) \mapsto (y_1, ..., y_n), \tag{10}$$

is a homeomorphism, where

$$y_k = \begin{cases} x_i & \text{if } k = i, \\ x_j & \text{if } k = j, \\ w_k x_i^{a_k} x_i^{b_k} & \text{if } k \neq i, j, \end{cases}$$

and $C(c)$ is the 2×2 antisymmetric matrix with entries in $\mathbb{Z}/2$ with $c_{12} = c_{21}^{-1} = c$. In addition, if $c \neq 1$, then $\mathcal{AC}_{SU(2)}(C(c)) \cong PU(2)$. It follows that all the path-connected components of

$B_n(SU(2),\mathbb{Z}/2)$ different from $\mathrm{Hom}(\mathbb{Z}^n, SU(2))$ are homeomorphic to $PU(2) \cong SO(3)$. Moreover, from the results in [5] it is easy to see that there are

$$D(n) := \frac{2^{n-2}(2^n-1)(2^{n-1}-1)}{3}$$

such connected components. Therefore

$$B_n(SU(2),\mathbb{Z}/2) \cong \left(\bigsqcup_{D(n)} PU(2)\right) \sqcup \mathrm{Hom}(\mathbb{Z}^n, SU(2)). \tag{11}$$

On the other hand, if $\underline{x} = (x_1, ..., x_n) \in \mathcal{AC}_{SU(2)}(C)$ is an almost commuting sequence in $SU(2)$ with $x_k = 1$ for some k, then either $\underline{x}$ is a commuting sequence in which case $\underline{x} \in S_n^1(SU(2))$ and C is trivial or C is a not trivial. If C is not trivial, then there are $1 \le i < j \le n$ such that $c := c_{i,j} \neq 1$. Note that necessarily $i, j \neq k$. Since $x_k = 1$, then $c_{ik} = [x_i, x_k] = 1$ and $c_{jk} = [x_j, x_k] = 1$, thus $a_k = b_k = 0$. It follows that in this case the k-th coordinate of the map ψ is given by $w_k x_i^{a_k} x_i^{b_k} = w_k \in \mathbb{Z}/2$. This shows that if $\underline{y} \in \mathcal{AC}_{SU(2)}(C)$ is in the same path-connected component as $\underline{x}$, then $y_k = 1$. It follows that $S_n^1(SU(2),\mathbb{Z}/2)$ is mapped under the homeomorphism in (11) onto the subspace

$$\left(\bigsqcup_{F(n)} PU(2)\right) \sqcup S_n^1(SU(2)),$$

where $F(n)$ is the number of path-connected components of $B_n(SU(2),\mathbb{Z}/2)$ homeomorphic to $PU(2)$ that contain an n-tuple $(x_1, ..., x_n)$ such that $x_k = 1$ for some k. Therefore

$$B_n(SU(2),\mathbb{Z}/2)/S_n^1(SU(2),\mathbb{Z}/2) \cong \mathrm{Hom}(\mathbb{Z}^n, SU(2))/S_n^1(SU(2)) \vee (\bigvee_{K(n)} PU(2)_+),$$

where the $K(n)$'s are integers that satisfy $K(1) = 0$, $K(2) = 1$ and

$$\sum_{r=1}^{n} K(r)\binom{n}{r} = \frac{2^{n-2}(2^n-1)(2^{n-1}-1)}{3}.$$

The solution of this recurrence equation is

$$K(n) = \frac{7^n}{24} - \frac{3^n}{8} + \frac{1}{12}.$$

Since $PU(2) \cong SO(3) \cong \mathbb{R}P^3$ then there are homeomorphisms

$$B_n(SU(2),\mathbb{Z}/2)/S_n^1(SU(2),\mathbb{Z}/2) \cong \begin{cases} \mathbb{S}^3 & \text{if } n = 1, \\ (\bigvee_{K(n)} \mathbb{R}P^3_+) \vee (\mathbb{R}P^2)^{n\lambda_2}/s_n(\mathbb{R}P^2) & \text{if } n \ge 2. \end{cases}$$

On the other hand, the situation modulo conjugation simplifies substantially. In this case, the action of $SU(2)$ on the connected components of $\bar{B}_n(SU(2),\mathbb{Z}/2)$ homeomorphic to $PU(2)$ is transitive. It follows that

$$\bar{B}_n(SU(2),\mathbb{Z}/2) \cong Z_n \sqcup \mathrm{Rep}(\mathbb{Z}^n, SU(2)),$$

where Z_n is a finite set with $D(n) = \frac{2^{n-2}(2^n-1)(2^{n-1}-1)}{3}$ points. This together with Theorem 6.1 shows that

$$\bar{B}_n(SU(2),\mathbb{Z}/2) \cong Z_n \sqcup (\mathbb{S}^1)^n/\Sigma_2,$$

with Σ_2 acting diagonally on $(\mathbb{S}^1)^n$ and by complex conjugation on each $\mathbb{S}^1$ factor and

$$\bar{B}_n(SU(2),\mathbb{Z}/2)/\bar{S}^1_n(SU(2),\mathbb{Z}/2) \cong (\bigvee_{K(n)} \mathbb{S}^0) \vee \mathbb{S}^n/\Sigma_2$$

where as before, if $\tau \in \Sigma_2 - \{1\}$ then for a point $(x_0, ..., x_n) \in \mathbb{S}^n$

$$\tau \cdot (x_0, ..., x_n) = (x_0, -x_1, ..., -x_n).$$

8. Stable factors modulo conjugation

In this section an explicit description for the stable factors $\bar{B}_r(G,K)/\bar{S}^1_r(G,K)$ is provided for $r = 1, 2$ and 3 and when G is a compact, simple, connected and simply connected Lie group. This is achieved using the work in [11].

Take G a compact, connected, simple and simply connected Lie group. Fix $T \subset G$ a maximal torus, let $\Phi = \Phi(T,G)$ be the root system associated to T, W the Weyl group and fix Δ a set of simple roots for Φ. For each $I \subset \Delta$ there is an associated torus S_I in G whose lie algebra is

$$\mathfrak{t}_I := \bigcap_{a\in I} \mathrm{Ker}(a) \subset \mathfrak{t}.$$

For each such I, let $L_I = DZ(S_I)$ be the derived group of the centralizer of the torus S_I in G. Also let $F_I = S_I \cap L_I$, this is a finite subgroup of the center of L_I. In [11] the spaces $\mathcal{M}_G(C) = \mathcal{AC}_G(C)/G$ are described in the following way. Given $\underline{x} \in \mathcal{M}_G(C)$, any maximal torus in $Z_G(\underline{x})$ is conjugated to a unique torus of the form S_I. The subset $I = I(\underline{x})$ only depends on $\underline{x}$ and it is a locally constant function of $\underline{x}$. Given $I \subset \Delta$, let $\mathcal{M}^I_G(C)$ be the subspace of conjugacy classes of almost commuting n-tuples in G of type C whose centralizer has a maximal torus conjugate to S_I. Therefore, each $\mathcal{M}^I_G(C) \subset \mathcal{M}_G(C)$ is a union of connected components. Moreover, by [11, Theorem 2.3.1] there is a homeomorphism

$$(S^n_I \times_{F^n_I} \mathcal{M}^0_{L_I}(C))/W(S_I,G) \cong \mathcal{M}^I_G(C) \tag{12}$$

where $\mathcal{M}^0_{L_I}(C)$ is the moduli space of almost commuting n-tuples $(x_1, ..., x_n)$ in G of type C for which $Z_G(x_1, ..., x_n)$ has rank 0 and $W(S_I,G)$ is the Weyl group of the torus S_I; that is, $W(S_I,G) = N_G(S_I)/Z_G(S_I)$. It follows that for $K \subset Z(G)$ a closed subgroup each $\bar{B}_n(G,K)$ is a union of spaces of the form (12). Moreover, in [11] explicit description for these spaces that only depend on the geometry of G are given for the cases $n = 1, 2$ and 3. These descriptions can be used to determine the stable factors $\bar{B}_r(G,K)/\bar{S}^1_r(G,K)$. To do this note that under the assumptions $Z(G)$ is finite and thus $\pi_0(K) = K$ for all $K \subset Z(G)$. Given $K \subset Z(G)$ a subgroup, by (1) there is a decomposition

$$B_r(G,K) = \bigsqcup_{C\in T(r,K)} \mathcal{AC}_G(C).$$

Each $\mathcal{AC}_G(C)$ is invariant under the action of G, thus on the level of orbit spaces

$$\bar{B}_r(G,K)=\bigsqcup_{C\in T(r,K)}\mathcal{M}_G(C),$$

where $\mathcal{M}_G(C)=\mathcal{AC}_G(C)/G$ is as defined before. Suppose that $(x_1,...,x_r)\in\mathcal{AC}_G(C)$ for some $C\in T(r,K)$. If $x_i=1_G$ for some $1\le i\le r$, then $c_{ij}=c_{ji}^{-1}=1_G$ for all $1\le j\le r$; that is, the elements in the i-th column and i-th row of C are all 1_G. For $0\le i\le r$ define $T_i(r,K)$ to be the subset of $T(r,K)$ consisting of matrices C that have exactly i rows (and hence i columns) whose entries are all 1_G. Note that if C is an $r\times r$ antisymmetric matrix with $(r-1)$ columns only having 1_G's, then all the entries of C are 1_G and thus $T_{r-1}(r,K)$ is empty. By definition

$$T(r,K)=\bigsqcup_{0\le i\le r}T_i(r,K)$$

and thus

$$B_r(G,K)=\bigsqcup_{0\le i\le r}\bigsqcup_{C\in T_i(r,K)}\mathcal{AC}_G(C).$$

In general, if $X=\sqcup_{\alpha\in\Lambda}X_\alpha$ and $A\subset X$ then $X/A=\bigvee_{\alpha\in\Lambda}X_\alpha/(A\cap X_\alpha)$, where the usual convention $X/\emptyset=X_+$ is used. In particular, with the notation

$$S_r^1(G,C):=S_r^1(G,K)\cap\mathcal{AC}_G(C)$$

there is an identification

$$B_r(G,K)/S_r^1(G,K)=\bigvee_{0\le i\le r}\bigvee_{C\in T_i(r,K)}\mathcal{AC}_G(C)/S_r^1(G,C).$$

When $i=r$ there is only one element in $T_r(r,\pi_0(K))$ which is the matrix whose entries are all 1_G that is denoted by C_{1_G}. In this case $\mathcal{AC}_G(C_{1_G})=\mathrm{Hom}(\mathbb{Z}^n,G)$ and $S_r^1(G,C_{1_G})=S_r^1(G)$, therefore

$$B_r(G,K)/S_r^1(G,K)=X_r(G,K)\bigvee\mathrm{Hom}(\mathbb{Z}^r,G)/S_r^1(G)$$

where

$$X_r(G,K)=\bigvee_{0\le i\le r-2}\bigvee_{C\in T_i(r,K)}\mathcal{AC}_G(C)/S_r^1(G,C)$$

From here it follows that all the stable factors in the decomposition of $\mathrm{Hom}(\mathbb{Z}^n,G)$ appear in the stable decomposition $B_n(G,K)$ for all closed subgroups $K\subset Z(G)$. On the level of orbit spaces

$$\bar{B}_r(G,K)/\bar{S}_r^1(G,K)=\bar{X}_r(G,K)\bigvee\mathrm{Rep}(\mathbb{Z}^r,G)/\bar{S}_r^1(G),\tag{13}$$

with

$$\bar{X}_r(G,K)=\bigvee_{0\le i\le r-2}\bigvee_{C\in T_i(r,K)}\mathcal{M}_G(C)/\bar{S}_r^1(G,C).$$

The cases $r=1,2$ and 3 are considered next.

- **Case r=1**. If $r=1$, then $B_1(G,K)=G$ and $S_1^1(G,K)=\{1_G\}$, thus

$$B_1(G,K)/S_1^1(G,K)=G,$$
$$\bar{B}_1(G,K)/\bar{S}_1^1(G,K)=G/G^{ad}=T/W.$$

- **Case r=2**. If $r = 2$ then by (13)

$$\bar{B}_2(G,K)/\bar{S}^1_2(G,K) = \bar{X}_2(G,K) \bigvee \mathrm{Rep}(\mathbb{Z}^2, G)/\bar{S}^1_2(G),$$

where

$$\bar{X}_2(G,K) = \bigvee_{C \in T_0(K)} \mathcal{M}_G(C)/\bar{S}^1_r(G,C).$$

Any matrix $C \in T(K)$ is of the form $C = C(c)$, where $C(c)$ denotes the 2×2 antisymmetric matrix with $c_{1,2} = c_{2,1}^{-1} = c \in K$. In particular any $C \in T_0(K)$ is of the form $C(c)$ for $c \in K - \{1_G\}$ and for such a matrix $S^1_2(G,C) = \emptyset$. Therefore

$$\bar{X}_2(G,K) = \bigvee_{c \in K - \{1_G\}} \mathcal{M}_G(C)/\bar{S}^1_r(G,C) = \bigvee_{c \in K - \{1_G\}} (\mathcal{M}_G(C))_+.$$

Each $\mathcal{M}_G(C)$ can be described in terms of the geometry of G as follows. Consider $\Phi^\vee$ the inverse root system and $\Delta^\vee$ the set of coroots $a^\vee$ inverse to roots $a \in \Delta$. Each $c \in K \subset Z(G)$ can be written in the form $c = \exp(\sum_{a \in \Delta} r_a a^\vee)$, where each $r_a \in \mathbb{Q}$. Given such c, following [11] define $I_c = \{a \in \Delta / r_a \notin \mathbb{Z}\} \subset \Delta$. By [11, Proposition 4.2.1] given $(x,y) \in G^2$ with $[x,y] = c$, then any maximal torus in $Z_G(x,y)$ is conjugate to S_{I_c}. In addition, by [11, Corollary 4.2.2]

$$\mathcal{M}_G(C(c)) \cong (\bar{S}_{I_c} \times \bar{S}_{I_c})/W(S_{I_c}, G).$$

On the other hand, in [10] Borel proved that for any pair x and y of commuting elements in a simply connected compact Lie group, there is a maximal torus that contains both x and y. Since all maximal tori are conjugate it follows that $\mathrm{Rep}(\mathbb{Z}^2, G) = (T \times T)/W$ and

$$\mathrm{Rep}(\mathbb{Z}^2, G)/\bar{S}^1_2(G) = (T \wedge T)/W.$$

Therefore

$$\bar{B}_2(G,K)/\bar{S}^1_2(G,K) = \left(\bigvee_{c \in K - \{1_G\}} ((\bar{S}_{I_c} \times \bar{S}_{I_c})/W(S_{I_c}, G))_+ \right) \vee (T \wedge T)/W$$

- **Case r=3**. The case $r = 3$ is similar. By (13)

$$\bar{B}_3(G,K)/\bar{S}^1_3(G,K) = \bar{X}_3(G,K) \bigvee \mathrm{Rep}(\mathbb{Z}^3, G)/\bar{S}^1_3(G)$$

with

$$\bar{X}_3(G,K) = \left(\bigvee_{C \in T_0(3,K)} \mathcal{M}_G(C)_+ \right) \vee \left(\bigvee_{C \in T_1(3,K)} \mathcal{M}_G(C)/\bar{S}^1_3(G,C) \right)$$

Using the work in [11] an explicit description for the terms in $\bar{X}_3(G,K)$ can be given. It can be shown that

$$\bar{B}_3(G,K)/\bar{S}^1_3(G,K) = Y_{3,0} \vee Y_{3,1} \vee Y_{3,2} \vee (T \wedge T \wedge T)/W$$

where

$$Y_{3,0} = \bigvee_{I_0} (\bar{S}^3_{I_0}/W(S_{I_0}, G))_+,$$
$$Y_{3,1} = \bigvee_{I_1} ((S^2_{I_1} \times (S_{I_1}/K_{I_1}))/W(S_{I_1}, G))_+,$$
$$Y_{3,2} = \bigvee_{I_2} (\bar{S}^2_{I_2} \times S_{I_2})/(\bar{S}^2_{I_2} \times \{1_G\})/W(S_{I_2}, G).$$

where I_0, I_1 and I_2 run through certain subsets of Δ that are determined uniquely by the geometry of G and K_{I_1} is a group acting on S_I of order at most 2. (In [11, Theorem 9.5.1] the different possibilities for K_{I_1} are discussed depending on the type of G).

In general, by [11, Theorem 2.3.1] the space $\bar{B}_n(G, Z(G))$ can be written as a disjoint union of spaces of the form

$$(S^n_I \times_{F^n_I} \mathcal{M}^0_{L_I}(C))/W(S_I, G). \tag{14}$$

It follows that for $n \geq 1$ the stable factor $\bar{B}_n(G, K)/\bar{S}^1_n(G, K)$ can be written as the wedge of certain quotients of spaces of the form (14).

References

[1] A. Adem. Z/pZ actions on $(S^n)^k$. Trans. Amer. Math. Soc. 300 (1987), no. 2, 791–809.

[2] A. Adem, A. Bahri, M. Bendersky, F. R. Cohen and S. Gitler. On decomposing suspension of simplicial spaces. Bol. Soc. Mat. Mex. (3) 15 (2009), 91102.

[3] A. Adem and F. R. Cohen. Commuting elements and spaces of homomorphisms. Math. Ann. 338 (2007), 587-626. Erratum. Math. Ann. 347 (2010), 245-248

[4] A. Adem, D. Cohen and F. R. Cohen. On Representations and K-theory of the Braid Groups. Math. Ann. 326 (2003), pp. 515–542.

[5] A. Adem, F. R. Cohen and J.M. Gómez. Commuting elements in central products of special unitary groups. Arxiv:math/0905.2895 [math.AT].

[6] A. Adem, F. R. Cohen and E. Torres-Giese. Commuting elements, simplicial spaces, and filtrations of classifying spaces. Arxiv:math/0901.0137 [math.AT].

[7] A. Adem, J. Ge, J. Pan and N. Petrosyan. Compatible actions and cohomology of crystallographic groups. J. Algebra 320 (2008), no. 1, 341–353.

[8] T. J. Baird. Cohomology of the space of commuting n-tuples in a compact Lie group. Algebraic & Geometric Topology, (2007), no.7 737–754,

[9] T. Baird, L. Jeffrey and P. Selick. The space of commuting n-tuples in SU(2). Arxiv:math:0911.4953 [math.AT]

[10] A. Borel. Sous-groupes commtatifs et torsion des groups de Lie compactes, Tohoku Math. Jour. 13(1962), 216-240.

[11] A. Borel, R. Friedman and J. W. Morgan. Almost commuting elements in compact Lie groups. Mem. Amer. Math. Soc. 157 (2002), no. 747.

[12] G. E. Bredon. Introduction to compact transformation groups. Pure and Applied Mathematics, Vol. 46. Academic Press, New York-London, 1972.

[13] M.C. Crabb. Spaces of commuting elements in $SU(2)$. Preprint.

[14] A. Dold and R. Thom. Quasifaserungen und unendliche symmetrische Produkte. Ann. of Math. (2) 67 1958 239–281.

[15] W. M. Goldman. Topological components of the space of representations. Invent Math. 93(3), 557-607 (1988).

[16] S. Helgason. Differential geometry, Lie groups, and symmetric spaces. Graduate Studies in Mathematics, American Mathematical Society, Providence, RI 2001.
[17] K.H. Hofmann and S.A. Morris. The structure of compact groups. A primer for the student—a handbook for the expert. de Gruyter Studies in Mathematics, 25. Walter de Gruyter & Co., Berlin, 1998. xviii+835 pp.
[18] I. G. Macdonald. Symmetric products of an algebraic curve. Topology 1 1962 319–343.
[19] J. P. May. The Geometry of Iterated Loop Spaces. Springer Lecture Notes in Mathematics. Springer, Berlin (1972).
[20] M. Murayama. On G-ANRs and their G-homotopy types. Osaka J. Math. 20 (1983), no. 3, 479–512.
[21] D. H. Park and D. Y. Suh. Equivariant semi-algebraic triangulations of real algebraic G-varieties. Kyushu J. Math. 50 (1996), no. 1, 179–205.
[22] D. Ramras. The stable moduli space of flat connections over a surface. arXiv:0810.1784v3 [math.AT].
[23] D. Sjerve and E. Torres-Giese, Fundamental groups of commuting elements in Lie groups. Bulletin of the London Mathematical Society 2008 40(1):65-76.
[24] N. E. Steenrod, A convenient category of topological spaces, Michigan Math. Journal 14 (1967), 133-152.
[25] N. E. Steenrod, Cohomology operations, and obstructions to extending continuous functions. Advances in Math. 8, 371–416. (1972).

Mathematics Department, University of British Columbia, Vancouver BC V6T1Z2, Canada
E-mail address: `adem@math.ubc.ca`

Department of Mathematics, University of Rochester, Rochester NY 14627, USA
E-mail address: `cohf@math.rochester.edu`

Mathematics Department, University of British Columbia, Vancouver BC V6T1Z2, Canada
E-mail address: `josmago@math.ubc.ca`